Logical Equivalences, Homomorphism Indistinguishability, and Forbidden Minors

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Abstract

Two graphs G and H are *homomorphism indistinguishable* over a class of graphs $\mathcal{F}$ if for all graphs $F \in \mathcal{F}$ the number of homomorphisms from F to G is equal to the number of homomorphisms from F to H. Many natural equivalence relations comparing graphs such as (quantum) isomorphism, spectral, and logical equivalences can be characterised as homomorphism indistinguishability relations over certain graph classes.

Abstracting from the wealth of such instances, we show in this paper that equivalences w.r.t. any *self-complementarity* logic admitting a characterisation as homomorphism indistinguishability relation can be characterised by homomorphism indistinguishability over a minor-closed graph class. Self-complementarity is a mild property satisfied by most well-studied logics. This result follows from a correspondence between closure properties of a graph class and preservation properties of its homomorphism indistinguishability relation.

Furthermore, we classify all graph classes which are in a sense finite (*essentially profinite*) and satisfy the maximality condition of being *homomorphism distinguishing closed*, i.e. adding any graph to the class strictly refines its homomorphism indistinguishability relation. Thereby, we answer various questions raised by Roberson (2022) on general properties of the homomorphism distinguishing closure.

1. Introduction

In 1967, Lovász [1] proved that two graphs G and H are isomorphic if and only if they are *homomorphism indistinguishable* over all graphs, i.e. for every graph F, the number of homomorphisms from F to G is equal to the number of homomorphisms from F to H. Since then, homomorphism indistinguishability over restricted graph classes has emerged as a powerful framework for capturing a wide range of equivalence relations comparing graphs. For example, two graphs have cospectral adjacency matrices iff they are homomorphism indistinguishable over all cycles, cf. [2]. They are quantum isomorphic iff they are homomorphism indistinguishable over all planar graphs [3].

Most notably, equivalences with respect to many logic fragments can be characterised as homomorphism indistinguishability relations over certain graph classes [4, 5, 6, 7]. For example, two graphs satisfy the same sentences of k-variable counting logic iff they are homomorphism indistinguishable over graphs of treewidth less than k [8]. All graph classes featured in such characterisations are minor-closed and hence of a particularly enjoyable structure. The main result of this paper asserts that this is not a mere coincidence: In fact, logical equivalences and homomorphism indistinguishability over minor-closed graph classes are intimately related.

To make this statement precise, the term 'logic' has to be formalised. Following [9], a *logic on graphs* is a pair $(\mathsf{L}, \models)$ of a class L of *sentences* and an isomorphism-invariant[2] *model relation* $\models$ between graphs and

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[1] Supported by the German Research Foundation (DFG) within Research Training Group 2236/2 (UnRAVeL) and the European Union (ERC, SymSim, 101054974).

[2] For all $\varphi \in \mathsf{L}$ and graphs G and H such that $G \cong H$, it holds that $G \models \varphi$ iff $H \models \varphi$.

Closure property of $\mathcal{F}$	Preservation property of $\equiv_{\mathcal{F}}$	Theorem
taking minors	complements	Theorem 8
taking summands	disjoint unions	Theorem 7
taking subgraphs	full complements	Theorem 13
taking induced subgraphs	left lexicographic products	Theorem 14
contracting edges	right lexicographic products	Theorem 15

Table 1: Overview of results on equivalent properties of a homomorphism distinguishing closed graph class $\mathcal{F}$ and of its homomorphism indistinguishability relation $\equiv_{\mathcal{F}}$.

sentences. Two graphs G and H are L-*equivalent* if $G \models \varphi$ iff $H \models \varphi$ for all $\varphi \in \mathsf{L}$. One may think of a logic on graphs as a collection of isomorphism-invariant graph properties. We call a logic *self-complementary* if for every $\varphi \in \mathsf{L}$ there is an element $\overline{\varphi} \in \mathsf{L}$ such that $G \models \varphi$ if and only if $\overline{G} \models \overline{\varphi}$. Here, $\overline{G}$ denotes the complement graph of G. Roughly speaking, a fragment/extension L of first-order logic is self-complementary if expressions of the form Exy can be replaced by $\neg Exy \wedge (x \neq y)$ in every formula while remaining in L. This lax requirement is satisfied by many logics including first-order logic, counting logic, second-order logic, fixed-point logics, and bounded variable, quantifier depth, or quantifier prefix fragments of these. All these examples are subject to the following result:

Theorem 1. *Let $(\mathsf{L}, \models)$ be a self-complementary logic on graphs for which there exists a graph class $\mathcal{F}$ such that simple graphs are homomorphism indistinguishable over $\mathcal{F}$ if and only if they are L-equivalent. Then there exists a minor-closed graph class $\mathcal{F}'$ whose homomorphism indistinguishability relation coincides with L-equivalence.*

Theorem 1 can be used to rule out that a given logic has a homomorphism indistinguishability characterisation (Corollary 19). Furthermore, it allows to use deep results of graph minor theory to study the expressive power of logics on graphs (Theorem 25).

Theorem 1 is product of a more fundamental study of the properties of homomorphism indistinguishability relations. In several instances, we show that closure properties of a graph class $\mathcal{F}$ correspond to preservation properties of its homomorphism indistinguishability relation $\equiv_{\mathcal{F}}$. These efforts yield answers to several open questions from [10]. A prototypical result is Theorem 2, from which Theorem 1 follows. For further results in the same vein, see Table 1.

Theorem 2. *Let $\mathcal{F}$ be a homomorphism distinguishing closed graph class. Then $\mathcal{F}$ is minor-closed if and only if $\equiv_{\mathcal{F}}$ is preserved under taking complements, i.e. for all simple graphs G and H it holds that $G \equiv_{\mathcal{F}} H$ if and only if $\overline{G} \equiv_{\mathcal{F}} \overline{H}$.*

Here, the graph class $\mathcal{F}$ is *homomorphism distinguishing closed* [10] if for every $K \notin \mathcal{F}$ there exist graphs G and H which are homomorphism indistinguishable over $\mathcal{F}$ but differ in the number of homomorphisms from K. In other words, adding even a single graph to $\mathcal{F}$ would change its homomorphism indistinguishability relation.

Proving that a graph class is homomorphism distinguishing closed is a pathway to separating equivalence relations comparing graphs [11]. However, establishing this property is a notoriously hard task. Thus, a general result establishing the homomorphism distinguishing closedness of a wide range of graph classes would be desirable. In [10], Roberson conjectured that *every graph class closed under taking minors and disjoint unions is homomorphism distinguishing closed*. At present, this conjecture has only been verified for few graph classes [10, 12].

Our final result confirms Roberson's conjecture for all graph classes which are in a certain sense finite. The expressive power of homomorphism counts from finitely many graphs is of particular importance in practice. Applications include the design of graph kernels [13], motif counting [14, 15], or machine learning on graphs [16, 17, 18]. A theoretical interest stems for example from database theory where homomorphism counts correspond to results of queries under bag-semantics [19, 20], see also [21].

Since every homomorphism distinguishing closed graph class is closed under taking disjoint unions, infinite graph classes arise inevitably when studying homomorphism indistinguishability over finite graph classes. We introduce the notions of *essentially finite* and *essentially profinite* graph classes (Theorem 26) in order to capture the limited behaviour of graph classes arising from finite graph classes. Examples for essentially profinite graph classes include the class of all minors of a fixed graph and the class of cluster graphs, i.e. disjoint unions of arbitrarily large cliques. In Theorem 27, the essentially profinite graph classes which are homomorphism distinguishing closed are fully classified. Thereby, the realm of available examples of homomorphism distinguishing closed graph classes is drastically enlarged. This classification has the following readily-stated corollary:

Theorem 3. *Every essentially profinite union-closed graph class $\mathcal{F}$ which is closed under taking summands*[3] *is homomorphism distinguishing closed. In particular, Roberson's conjecture holds for all essentially profinite graph classes.*

2. Preliminaries

All graphs in this article are finite, undirected, and without multiple edges. A *simple graph* is a graph without loops. A *homomorphism* from a graph F to a graph G is a map $h\colon V(F) \to V(G)$ such that $h(u)h(v) \in E(G)$ whenever $uv \in E(F)$ and vertices carrying a loop are mapped to vertices carrying a loop. Write $\hom(F,G)$ for the number of homomorphisms from F to G. For a class of graphs $\mathcal{F}$ and graphs G and H, write $G \equiv_{\mathcal{F}} H$ if $\hom(F,G) = \hom(F,H)$ for all $F \in \mathcal{F}$, i.e. G and H are *homomorphism indistinguishable over* $\mathcal{F}$. With the exception of Section 3.3, the graphs in $\mathcal{F}$ as well as G and H will be simple. Following [10], the *homomorphism distinguishing closure* of $\mathcal{F}$ is

$$\operatorname{cl}(\mathcal{F}) \coloneqq \{K \text{ simple graph} \mid \forall \text{ simple graphs } G, H.\quad G \equiv_{\mathcal{F}} H \Rightarrow \hom(K,G) = \hom(K,H)\}.$$

Intuitively, $\operatorname{cl}(\mathcal{F})$ is the "largest" graph class whose homomorphism indistinguishability relation coincides with the one of $\mathcal{F}$. A graph class $\mathcal{F}$ is *homomorphism distinguishing closed* if $\operatorname{cl}(\mathcal{F}) = \mathcal{F}$. Note that cl is a closure operator in the sense that $\operatorname{cl}(\mathcal{F}) \subseteq \operatorname{cl}(\mathcal{F}')$ if $\mathcal{F} \subseteq \mathcal{F}'$ and $\operatorname{cl}(\operatorname{cl}(\mathcal{F})) = \operatorname{cl}(\mathcal{F})$ for all graph classes $\mathcal{F}$ and $\mathcal{F}'$. Furthermore, the following inclusions hold (and cannot be reversed, cf. Example 34):

Lemma 4. *Let I be an arbitrary index set and let $(\mathcal{F}_i)_{i\in I}$ be a family of graph classes. Then*

$$\operatorname{cl}\left(\bigcap_{i\in I} \mathcal{F}_i\right) \subseteq \bigcap_{i\in I} \operatorname{cl}(\mathcal{F}_i) \quad \textit{and} \quad \bigcup_{i\in I} \operatorname{cl}(\mathcal{F}_i) \subseteq \operatorname{cl}\left(\bigcup_{i\in I} \mathcal{F}_i\right).$$

Proof. Any intersection of homomorphism distinguishing closed graph classes is homomorphism distinguishing closed [10, Lemma 6.1]. Hence, $\bigcap_{i\in I} \operatorname{cl}(\mathcal{F}_i)$ is closed and it suffices to observe that $\bigcap_{i\in I} \mathcal{F}_i \subseteq \bigcap_{i\in I} \operatorname{cl}(\mathcal{F}_i)$. For the second claim, $\mathcal{F}_j \subseteq \bigcup_{i\in I} \mathcal{F}_i$ and hence $\operatorname{cl}(\mathcal{F}_j) \subseteq \operatorname{cl}(\bigcup_{i\in I} \mathcal{F}_i)$ for $j \in I$. ☐

For graphs F and K, F is said to be *K-colourable* if there is a homomorphisms $F \to K$. F and K are *homomorphically equivalent* if there are homomorphisms $F \to K$ and $K \to F$.

For graphs G and H, write $G + H$ for their *disjoint union* and $\coprod_{i\in[n]} G_i \coloneqq G_1 + \cdots + G_n$ for graphs $G_1, \dots, G_n$. We write $G \times H$ for the *categorical product* of G and H, i.e. $V(G \times H) \coloneqq V(G) \times V(H)$ and gh and $g'h'$ are adjacent in $G \times H$ iff $gg' \in E(G)$ and $hh' \in E(H)$. The *lexicographic product* $G \cdot H$ is defined as the graph with vertex set $V(G) \times V(H)$ and edges between gh and $g'h'$ iff $g = g'$ and $hh' \in E(H)$ or $gg' \in E(G)$. It is well-known, cf. e.g. [22, (5.28)–(5.30)], that for all graphs F_1, F_2, G_1, G_2, and all connected graphs K, the following equalities hold:

$$\hom(F_1 + F_2, G) = \hom(F_1, G)\hom(F_2, G), \tag{1}$$
$$\hom(F, G_1 \times G_2) = \hom(F, G_1)\hom(F, G_2), \text{ and} \tag{2}$$
$$\hom(K, G_1 + G_2) = \hom(K, G_1) + \hom(K, G_2). \tag{3}$$

[3] A graph class $\mathcal{F}$ is *closed under taking summands* if for every $F \in \mathcal{F}$ which is the disjoint union of two graphs $F_1 + F_2 = F$ also $F_1, F_2 \in \mathcal{F}$.

The *complement* of a simple graph F is the simple graph $\overline{F}$ with $V(\overline{F}) = V(F)$ and $E(\overline{F}) = \binom{V(F)}{2} \setminus E(F)$. Observe that $\overline{\overline{F}} = F$ for all simple graphs F. The *full complement* of a graph G is the graph $\widehat{G}$ obtained from G by replacing every edge with a non-edge and every loop with a non-loop, and vice-versa.

The *quotient* $F/\mathcal{P}$ of a simple graph F by a partition $\mathcal{P}$ of $V(F)$ is the simple graph with vertex set $\mathcal{P}$ and edges PQ for $P \neq Q$ iff there exist vertices $p \in P$ and $q \in Q$ such that $pq \in E(F)$. For a set X, write $\Pi(X)$ for the set of all partitions of X. A graph F' can be obtained from a simple graph F by *contracting edges* if there is a partition $\mathcal{P} \in \Pi(V(F))$ such that $F[P]$ is connected for all $P \in \mathcal{P}$ and $F' \cong F/\mathcal{P}$.

For a graph F and a set $P \subseteq V(F)$, write $F[P]$ for the *subgraph induced by* P, i.e. the graph with vertex set P and edges uv if $u, v \in P$ and $uv \in E(F)$. A graph F' is a *subgraph* of F, in symbols $F' \subseteq F$ if $V(F') \subseteq V(F)$ and $E(F') \subseteq E(F)$. A *minor* of a simple graph F is a subgraph of a graph which can be obtained from F by contracting edges.

Let F be a simple graph with edge $uv \in E(F)$. The graph obtained from F by *subdividing* uv is the graph F' with vertex set $V(F') \coloneqq V(F) \sqcup \{w\}$ and $E(F') \coloneqq (E(F) \setminus \{uv\}) \cup \{uw, wv\}$. A graph F' is a *subdivision* of a simple graph F if it can be obtained from F by repeatedly subdividing edges.

Fact 5 (Inclusion–Exclusion). *Let $A_1, \dots, A_n$ be subsets of a finite set S. For $A \subseteq S$, write $\overline{A} \coloneqq S \setminus A$. Then*

$$\left| \bigcap_{i \in [n]} \overline{A_i} \right| = |S| + \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|.$$

3. Closure Properties Correspond to Preservation Properties

This section is concerned with the interplay of closure properties of a graph class $\mathcal{F}$ and preservation properties of its homomorphism indistinguishability relation $\equiv_{\mathcal{F}}$. The central results of this section are Theorem 2 and the other results listed in Table 1.

The relevance of the results is twofold: On the one hand, they yield that if a graph class $\mathcal{F}$ has a certain closure property then so does $\mathrm{cl}(\mathcal{F})$. In the case of minor-closed graph families, this provides evidence for Roberson's conjecture [10]. On the other hand, they establish that equivalence relations comparing graphs which are preserved under certain operations coincide with the homomorphism indistinguishability relation over a graph class with a certain closure property, if they are homomorphism indistinguishability relations at all. Further consequences are discussed in Sections 3.5 and 4. Essential to all proofs is the following Lemma 6:

Lemma 6. *Let $\mathcal{F}$ and $\mathcal{L}$ be classes of simple graphs. Suppose $\mathcal{L}$ is finite and that its elements are pairwise non-isomorphic. Let $\alpha \colon \mathcal{L} \to \mathbb{R} \setminus \{0\}$. If for all simple graphs G and H*

$$G \equiv_{\mathcal{F}} H \implies \sum_{L \in \mathcal{L}} \alpha_L \hom(L, G) = \sum_{L \in \mathcal{L}} \alpha_L \hom(L, H)$$

then $\mathcal{L} \subseteq \mathrm{cl}(\mathcal{F})$.

Proof. The following argument is due to [23, Lemma 3.6]. Let n be an upper bound on the number of vertices of graphs in $\mathcal{L}$ and let $\mathcal{L}'$ denote the class of all graphs on at most n vertices. By classical arguments [1], cf. [22, Proposition 5.44(b)], the matrix $M \coloneqq (\hom(K, L))_{K,L \in \mathcal{L}'}$ is invertible. Extend α to a function $\alpha' \colon \mathcal{L}' \to \mathbb{R}$ by setting $\alpha'(L) \coloneqq \alpha(L)$ for all $L \in \mathcal{L}$ and $\alpha'(L') \coloneqq 0$ for all $L' \in \mathcal{L}' \setminus \mathcal{L}$. By Eq. (2), if $G \equiv_{\mathcal{F}} H$ then $G \times K \equiv_{\mathcal{F}} H \times K$ for all graphs K. Hence, by Eq. (2),

$$\sum_{L \in \mathcal{L}'} \hom(L, K) \alpha_L \hom(L, G) = \sum_{L \in \mathcal{L}'} \hom(L, K) \alpha_L \hom(L, H).$$

Both sides can be read as the product of the matrix M^T with a vector of the form $(\alpha_L \hom(L, -))_{L \in \mathcal{L}'}$. By multiplying from the left with the inverse of M^T, it follows that $\alpha_L \hom(L, G) = \alpha_L \hom(L, H)$ for all $L \in \mathcal{L}'$ which in turn implies that $\hom(L, G) = \hom(L, H)$ for all $L \in \mathcal{L}$. Thus, $\mathcal{L} \subseteq \mathrm{cl}(\mathcal{F})$. $\square$

$$\begin{tikzcd}
 & \text{edge contractions} \arrow[rrr, dashed] & & & \text{summands} \\
\text{minors} \arrow[ur] \arrow[dr] & & & & \\
 & \text{edge deletion} \arrow[r, dashed, leftrightarrow] & \text{subgraphs} \arrow[r] & \text{induced subgraphs} \arrow[uur] &
\end{tikzcd}$$

Figure 1: Relationship between closure properties of homomorphism distinguishing closed graph classes. The non-obvious implications are dashed and proven in Lemmas 12 and 18.

In the setting of Lemma 6, we say that the relation $\equiv_{\mathcal{F}}$ *determines* the linear combination $\sum_{L \in \mathcal{L}} \alpha_L \hom(L, -)$. Note that it is essential for the argument to carry through that the elements of $\mathcal{L}$ are pairwise non-isomorphic and that $\alpha_L \neq 0$ for all L. Efforts will be undertaken to establish this property for certain linear combinations in the subsequent sections.

3.1. Taking Summands and Preservation under Disjoint Unions

In this section, the strategy yielding the results in Table 1 is presented for the rather simple case of Theorem 7. This theorem relates the property of a graph class $\mathcal{F}$ to be closed under taking summands to the property of $\equiv_{\mathcal{F}}$ to be preserved under disjoint unions. This closure property is often assumed in the context of homomorphism indistinguishability [24, 10] and fairly mild. It is the most general property among those studied here, cf. Fig. 1. Theorem 7 answers a question from [10, p. 7] affirmatively: Is it true that if $\equiv_{\mathcal{F}}$ is preserved under disjoint unions then $\mathrm{cl}(\mathcal{F})$ is closed under taking summands?

Theorem 7. *For a graph class $\mathcal{F}$ and the assertions*

1. $\mathcal{F}$ *is* closed under taking summands, *i.e. if $F_1 + F_2 \in \mathcal{F}$ then $F_1, F_2 \in \mathcal{F}$,*
2. $\equiv_{\mathcal{F}}$ *is* preserved under disjoint unions, *i.e. for all simple graphs G, G', H, and H', if $G \equiv_{\mathcal{F}} G'$ and $H \equiv_{\mathcal{F}} H'$ then $G + H \equiv_{\mathcal{F}} G' + H'$,*
3. $\mathrm{cl}(\mathcal{F})$ *is closed under taking summands,*

the implications 1 $\Rightarrow$ 2 $\Leftrightarrow$ 3 hold.

Proof. The central idea is to write, given graphs F, G, and H, the quantity $\hom(F, G+H)$ as expression in $\hom(F', G)$ and $\hom(F', H)$ where the F' range over summands of F. To this end, write $F = C_1 + \dots + C_r$ as disjoint union of its connected components. Then,

$$\begin{aligned} \hom(F, G+H) &\overset{(1)}{=} \prod_{i=1}^{r} \hom(C_i, G+H) \overset{(3)}{=} \prod_{i=1}^{r} (\hom(C_i, G) + \hom(C_i, H)) \\ &\overset{(1)}{=} \sum_{I \subseteq [r]} \hom(\coprod_{i \in I} C_i, G) \hom(\coprod_{i \in [r] \setminus I} C_i, H). \end{aligned} \tag{4}$$

In particular, if $\mathcal{F}$ is closed under taking summands then $\coprod_{i \in I} C_i \in \mathcal{F}$ for all $I \subseteq [r]$. Thus, 1 implies 2.

Assume 2 and let $F \in \mathrm{cl}(\mathcal{F})$. Write as above $F = C_1 + \dots + C_r$ as disjoint union of its connected components. By the assumption that $\equiv_{\mathcal{F}}$ is preserved under disjoint unions, for all graphs G and G', if $G \equiv_{\mathcal{F}} G'$ then $G + F \equiv_{\mathcal{F}} G' + F$ and hence $\hom(F, G+F) = \hom(F, G'+F)$. By Eq. (4) with $H = F$, the relation $\equiv_{\mathcal{F}}$ determines the linear combination $\sum_{I \subseteq [r]} \hom(\coprod_{i \in I} C_i, -) \hom(\coprod_{i \in [r] \setminus I} C_i, F)$. Note that it might be the case that $\coprod_{i \in I} C_i \cong \coprod_{j \in J} C_j$ for some $I \neq J$. Grouping such summands together and adding their coefficients yields a linear combination satisfying the assumptions of Lemma 6 since $\hom(C_i, F) > 0$ for all $i \in [r]$. Hence, $\coprod_{i \in I} C_i \in \mathrm{cl}(\mathcal{F})$ for all $I \subseteq [r]$ and 3 follows.

The implication 3 $\Rightarrow$ 2 follows from 1 $\Rightarrow$ 2 for $\mathrm{cl}(\mathcal{F})$ since $\equiv_{\mathcal{F}}$ and $\equiv_{\mathrm{cl}(\mathcal{F})}$ coincide. $\square$

The proofs of the other results in Table 1 are conceptually similar to the just completed proof. The general idea can be briefly described as follows:

1. Derive a linear expression similar to Eq. (4) for the number of homomorphisms from F into the graph constructed using the assumed preservation property of $\equiv_{\mathcal{F}}$, e.g. the graph $G + H$ in the case of Theorem 7. These linear combinations typically involve sums over subsets U of vertices or edges of F, each contributing a summand of the form $\alpha_U \hom(F_U, -)$ where α_U is some coefficient and F_U is a graph constructed from F using U. Hence, if $\mathcal{F}$ is closed under the construction transforming F to F_U then $\equiv_{\mathcal{F}}$ has the desired preservation property.
2. In general, it can be that F_U and $F_{U'}$ are isomorphic even though $U \neq U'$, e.g. in Eq. (4) if F contains two isomorphic connected components. In order to apply Lemma 6, one must group the summands $\alpha_U \hom(F_U, -)$ by the isomorphism type F' of the F_U. The coefficient of $\hom(F', -)$ in the new linear combination ranging over pairwise non-isomorphic graphs is the sum of α_U over all U such that $F_U \cong F'$. Once it is established that this coefficient is non-zero, it follows that if $\equiv_{\mathcal{F}}$ has the preservation property then $\mathrm{cl}(\mathcal{F})$ has the desired closure property.

3.2. Taking Minors and Preservation under Complements

The strategy outlined in the previous section is now applied to prove Theorem 8, which implies Theorem 2. This answers a question of Roberson [10, Question 8] affirmatively: Is it true that if $\mathcal{F}$ is such that $\equiv_{\mathcal{F}}$ is preserved under taking complements then there exist a minor-closed $\mathcal{F}'$ such that $\equiv_{\mathcal{F}}$ and $\equiv_{\mathcal{F}'}$ coincide.

Theorem 8 is among the first results substantiating Roberson's conjecture not only for example classes but in full generality. In particular, as noted in [10, p. 2], there was little reason to believe that minor-closed graph families should play a distinct role in the theory of homomorphism indistinguishability. Theorem 8 indicates that this might be the case. Indeed, while Roberson's conjecture asserts that $\mathrm{cl}(\mathcal{F})$ coincides with $\mathcal{F}$ for every minor-closed and union-closed graph class $\mathcal{F}$, Theorem 8 yields unconditionally that $\mathrm{cl}(\mathcal{F})$ is a minor-closed graph class itself.

Theorem 8. *For a graph class $\mathcal{F}$ and the assertions*

1. *$\mathcal{F}$ is closed under edge contraction and deletion,*
2. *$\equiv_{\mathcal{F}}$ is* preserved under taking complements*, i.e. for all simple graphs G and H it holds that $G \equiv_{\mathcal{F}} H$ if and only if $\overline{G} \equiv_{\mathcal{F}} \overline{H}$,*
3. *$\mathrm{cl}(\mathcal{F})$ is minor-closed,*

the implications $1 \Rightarrow 2 \Leftrightarrow 3$ hold.

The strategy is to write $\hom(F, \overline{G})$ as a linear combination of $\hom(F', G)$ for minors F' of F. This is accomplished in two steps: First, we consider the full complement $\widehat{G}$ of G in which not only every edge of G is replaced by a non-edge (and vice versa) but also every loop is replaced by a non-loop (and vice versa). Secondly, we consider for a simple graph G the looped graph G° obtained from G by adding a loop to every vertex. The fact that $\widehat{G^\circ} \cong \overline{G}$ motivates this two-step approach.

Lemma 9 ([22, Equation (5.23)])**.** *For every graph G and every simple graph F,*

$$\hom(F, \widehat{G}) = \sum_{F' \subseteq F \text{ s.t. } V(F') = V(F)} (-1)^{|E(F')|} \hom(F', G).$$

Proof. Write S for the set of all maps $V(F) \to V(G)$. For $e \in E(F)$, write $A_e \subseteq S$ for set of all maps h such that the image of e under h is an edge or a loop in G. It holds that $h \notin A_e$ if and only if e is mapped under h either to two non-adjacent vertices or to a single vertex without a loop. Hence, $\hom(F, \widehat{G}) = \left|\bigcap_{e \in E(F)} \overline{A_e}\right|$. The claim follows from the Inclusion–Exclusion Principle (cf. Fact 5) noting that $|S| = \hom(|F|K_1, G)$ and $\left|\bigcap_{e \in F'} A_e\right|$ for some $F' \subseteq F$ is the number of homomorphisms from the graph F' to G which is obtained by deleting all edges from F which are not in F'. $\square$

In light of Lemma 9, it suffices to write $\hom(F, G^\circ)$ as a linear combination of $\hom(F', G)$ for minors F' of F. To ease bookkeeping, we consider a particular type of quotient graphs. For a simple graph F and a set of edges $L \subseteq E(F)$, define the *contraction relation* $\sim_L$ on $V(F)$ by declaring $v \sim_L w$ if v and w lie in the same connected component of the subgraph of F with vertex set $V(F)$ and edge set L. Write $[v]_L$ for the classes of $v \in V(F)$ under the equivalence relation $\sim_L$.

The *contraction quotient* $F \oslash L$ is the graph whose vertex set is the set of equivalence classes under $\sim_L$ and with an edge between $[v]_L$ and $[w]_L$ if and only if there is an edge $xy \in E(F) \setminus L$ such that $x \sim_L v$ and $y \sim_L w$. In general, $F \oslash L$ may contain loops, cf. Example 11. However, if it is simple then it is equal to $F/\mathcal{P}$ where $\mathcal{P}$ is the partition of $V(F)$ into equivalence classes under $\sim_L$, i.e. $\mathcal{P} \coloneqq \{[v]_L \mid v \in V(F)\}$. In this case, $F \oslash L$ is a graph obtained from F by edge contractions.

With this notation, the quantity $\hom(F, G^\circ)$ can be succinctly written as a linear combination. Using Lemma 9 and Theorem 10, Theorem 8 can be proven by the strategy described in Section 3.1.

Theorem 10. *Let F and G be simple graphs. Then*

$$\hom(F, G^\circ) = \sum_{L \subseteq E(F)} \hom(F \oslash L, G).$$

Proof. The proof is by establishing a bijection between the set of homomorphisms $h \colon F \to G^\circ$ and the set X of pairs (L, φ) where $L \subseteq E(F)$ and $\varphi \colon F \oslash L \to G$ is a homomorphism.

Claim 10.1. *The map which associates a homomorphism $h \colon F \to G^\circ$ with the pair $(L, \varphi) \in X$ consisting of the set $L \coloneqq \{e \in E(F) \mid |h(e)| = 1\}$ and the homomorphism $\varphi \colon F \oslash L \to G$, $[x]_L \mapsto h(x)$ is well-defined.*

Proof of Claim. First observe that if $v, w \in V(F)$ are such that $v \sim_L w$ then $h(v) = h(w)$. Indeed, if wlog $v \neq w$ then there exists a walk $vu_1, u_1u_2, \dots, u_nw \in L$. By construction, $|h(vu_1)| = |h(u_1u_2)| = \cdots = |h(u_nw)| = 1$ and hence $h(v) = h(w)$. In particular, $F \oslash L$ is a simple graph without any loops. Indeed, if $vw \in E(F)$ are such that $v \sim_L w$ then $h(v) = h(w)$ and hence $vw \in L$.

The initial observation implies that φ is a well-defined map $V(F \oslash L) \to V(G)$. It remains to argue that φ is a homomorphism $F \oslash L \to G$. Indeed, if $[v]_L \neq [w]_L$ are adjacent in $F \oslash L$ then there exist $x \sim_L v$ and $y \sim_L w$ such that $xy \in E(F) \setminus L$. Hence, h maps xy to an edge in G°, rather than to a loop. In particular, φ maps $[v]_L$ and $[w]_L$ to an edge in G. ⌟

Claim 10.2. *The map $h \mapsto (L, \varphi)$ described in Claim 10.1 is surjective onto X.*

Proof of Claim. Let $L \subseteq E(F)$ and $\varphi \colon F \oslash L \to G$ be a homomorphism. Observe that this implies that $F \oslash L$ is without loops. Let $\pi \colon V(F) \to V(F \oslash L)$ denote the projection $v \mapsto [v]_L$. It is claimed that $h \coloneqq \varphi \circ \pi$ is a homomorphism $F \to G^\circ$. Let $xy \in E(F)$. If $x \sim_L y$ then the image of xy under h is a loop since then $\pi(x) = \pi(y)$. If $x \not\sim_L y$ then in particular $xy \notin L$ and there is an edge between $[x]_L$ and $[y]_L$ in $F \oslash L$. Since φ is a homomorphism, $h(x)$ and $h(y)$ are in both cases adjacent in G°.

It remains to argue that this h is mapped to (L, φ) under the construction described in Claim 10.1. Write $L' \coloneqq \{e \in E(F) \mid |h(e)| = 1\}$. Towards concluding that $L = L'$, let $uv \in L$. Then $h(uv) = \varphi(\pi(uv))$ is a singleton since in particular $u \sim_L v$. Hence, $L \subseteq L'$. Conversely, let $uv \in L'$. By assumption, $h(uv) = \varphi(\pi(uv))$ is a singleton and thus it remains to distinguish two cases: If $\pi(u) = \pi(v)$ then $uv \in L$ because otherwise there would be a loop at $[u]_L = [v]_L$ in $F \oslash L$. If $\pi(u) \neq \pi(v)$ then $\varphi([u]_L) = \varphi([v]_L)$ and also $uv \in L$ because otherwise there would be an edge between $[u]_L \neq [v]_L$ in $F \oslash L$ and this cannot happen since φ is a homomorphism into a simple graph.

Finally, write φ' for the homomorphism $F \oslash L \to G$ constructed from h as described in Claim 10.1. Then for every vertex $x \in V(F)$ by definition, $\varphi'([x]_L) = h(x) = \varphi(\pi(x)) = \varphi([x]_L)$ and thus $\varphi = \varphi'$. ⌟

It is easy to see that the map devised in Claim 10.1 is injective. The desired equation follows observing that the map in Claim 10.1 provides a bijection between the sets whose elements are counted on the right and left hand-side respectively. □

The following Example 11 illustrates the above construction and Theorem 10.

Example 11. *Let K_3 denote the clique with vertex set $\{1, 2, 3\}$. Then $K_3 \oslash \emptyset \cong K_3$, $K_3 \oslash \{12\} \cong K_2$, $K_3 \oslash \{12, 23\} \cong K_1^\circ$, and $K_3 \oslash \{12, 23, 13\} \cong K_1$. For every simple graph G,* $\hom(K_3, G^\circ) = \hom(K_3, G) + 3\hom(K_2, G) + \hom(K_1, G)$ *since* $\hom(K_1^\circ, G) = 0$.

The final ingredient for the proof of Theorem 8 is the following Lemma 12, which establishes one of the implications in Fig. 1.

Lemma 12. *If a homomorphism distinguishing closed graph class $\mathcal{F}$ is closed under deleting edges then it is closed under taking subgraphs.*

Proof. Let $F \in \mathcal{F}$ be a graph on n vertices. Since the graph nK_1 can be obtained from F by deleting all its edges, it holds that $nK_1 \in \mathcal{F}$. Thus, for all graphs G and H such that $G \equiv_{\mathcal{F}} H$, by Eq. (1), $|V(G)|^n = \hom(nK_1, G) = \hom(nK_1, H) = |V(H)|^n$. Hence, $\hom(K_1, G) = \hom(K_1, H)$. Now let F' denote the graph obtained from F by deleting a vertex $v \in V(F)$. By deleting all incident edges, it can be assumed that v is isolated in F and that $F' + K_1 \cong F$. Hence, for all G and H as above,

$$\hom(F', G)\hom(K_1, G) = \hom(F, G) = \hom(F, H) = \hom(F', H)\hom(K_1, H).$$

It follows that $\hom(F', G) = \hom(F', H)$ and $F' \in \mathcal{F}$. $\square$

Proof of Theorem 8. Assuming 1, let G and H be such that $G \equiv_{\mathcal{F}} H$ and let $F \in \mathcal{F}$. By Theorem 10 and Lemma 9,

$$\hom(F, \overline{G}) = \hom(F, \widehat{G^\circ}) = \sum_{\substack{F' \subseteq F,\\ V(F')=V(F)}} (-1)^{|E(F')|} \sum_{L \subseteq E(F')} \hom(F' \oslash L, G). \tag{5}$$

All simple graphs $F' \oslash L$ appearing in this sum are obtained from F by repeated edge contractions or deletions. Hence, $\overline{G} \equiv_{\mathcal{F}} \overline{H}$.

Assuming 2, it is first shown that $\mathrm{cl}(\mathcal{F})$ is closed under deleting and contracting edges. To that end, let $F \in \mathrm{cl}(\mathcal{F})$. Let K be obtained from F by deleting an edge $e \in E(F)$. Since K has the same number of vertices as F and precisely one edge less, the only summation indices (F', L) in Eq. (5) such that $F' \oslash L \cong K$ satisfy $|E(F')| = |E(K)|$ and $L = \emptyset$. Hence, the coefficient of $\hom(K, -)$ in the linear combination obtained from Eq. (5) by grouping isomorphic summation indices is a non-zero multiple of $(-1)^{|E(K)|}$ and $K \in \mathrm{cl}(\mathcal{F})$ by Lemma 6.

Let now K be obtained from F by contracting a single edge e. By deleting edges, it can be supposed without loss of generality that there are no vertices in F which are adjacent to both end vertices of e. By the following Claim 12.1, $|E(K)| = |E(F)| - 1$.

Claim 12.1. *For a simple graph F with $uv \in E(F)$, write $\Delta_F(uv) := \{w \in V(F) \mid vw, uw \in E(F)\}$ for the set of vertices inducing a triangle with the edge uv. Let F be a simple graph and $uv \in E(F)$. Then*

$$|\Delta_F(uv)| = |E(F)| - |E(F \oslash \{uv\})| - 1.$$

Proof of Claim. Write $K = F \oslash \{uv\}$. By writing x for the vertex of K obtained by contraction and identifying $V(K) \setminus \{x\} = V(F) \setminus \{u, v\}$, for all $w \in V(K)$,

$$\deg_K w = \begin{cases} \deg_F u + \deg_F v - |\Delta_F(uv)| - 2, & \text{if } w = x,\\ \deg_F w - 1, & \text{if } w \in \Delta_F(uv),\\ \deg_F w, & \text{otherwise.} \end{cases}$$

The desired equation now follows readily from the Handshaking Lemma. ⌟

Hence, all summation indices (F', L) in Eq. (5) such that $F' \oslash L \cong K$ must satisfy the following:

1. $L = \{e'\}$ for some edge $e' \in E(F)$.
   This is immediate from $|V(K)| = |V(F)| - 1$ and $V(F) = V(F')$.
2. $F' = F$.
   Indeed, if $F' \oslash \{e'\} \cong K$ for some $F' \subseteq F$ with $e' \in E(F')$ and $V(F') = V(F)$ then, by Claim 12.1, $|E(F')| - 1 = |E(F' \oslash \{e'\})| + |\Delta_{F'}(e')| \geq |E(F' \oslash \{e'\})| = |E(K)| = |E(F)| - 1$ and thus $E(F') = E(F)$.

Each of these summation indices contributes $(-1)^{|E(F')|} = (-1)^{|E(F)|}$ to the coefficient of $\hom(K, -)$ in the linear combination ranging over non-isomorphic graphs. Hence, this coefficient is non-zero. By Lemma 6, $K \in \operatorname{cl}(\mathcal{F})$.

Hence, $\operatorname{cl}(\mathcal{F})$ is closed under deleting and contracting edges. By Lemma 12, it is closed under taking minors. The implication 3 ⇒ 2 follows from 1 ⇒ 2 for $\operatorname{cl}(\mathcal{F})$ since $\equiv_{\mathcal{F}}$ and $\equiv_{\operatorname{cl}(\mathcal{F})}$ coincide. □

3.3. Taking Subgraphs and Preservation under Full Complements

Theorem 13, which relates the property of a graph class $\mathcal{F}$ to be closed under taking subgraphs to the property of $\equiv_{\mathcal{F}}$ to be preserved under taking full complements, can now be extracted from the insights in Section 3.2. Since our definition of the homomorphism distinguishing closure involves only simple graphs in order to be aligned with [10], Theorem 13 deviates slightly from the other results in Table 1. This is because the relations $\equiv_{\mathcal{F}}$ and $\equiv_{\operatorname{cl}(\mathcal{F})}$ a priori coincide only on simple graphs and not necessarily on all graphs, a crucial point raised by a reviewer.

Theorem 13. *For a graph class $\mathcal{F}$ and the assertions*

1. *$\mathcal{F}$ is closed under deleting edges,*
2. *$\equiv_{\mathcal{F}}$ is* preserved under taking full complements of arbitrary graphs*, i.e. for all graphs G and H it holds that $G \equiv_{\mathcal{F}} H$ if and only if $\widehat{G} \equiv_{\mathcal{F}} \widehat{H}$,*
3. *$\equiv_{\mathcal{F}}$ is* preserved under taking full complements*, i.e. for all simple graphs G and H it holds that $G \equiv_{\mathcal{F}} H$ if and only if $\widehat{G} \equiv_{\mathcal{F}} \widehat{H}$,*
4. *$\operatorname{cl}(\mathcal{F})$ is closed under deleting edges,*
5. *$\operatorname{cl}(\mathcal{F})$ is* closed under taking subgraphs*, i.e. it is closed under deleting edges and vertices,*
6. *$\equiv_{\operatorname{cl}(\mathcal{F})}$ is preserved under taking full complements of arbitrary graphs,*
7. *$\equiv_{\operatorname{cl}(\mathcal{F})}$ is preserved under taking full complements,*

the implications 1 ⇒ 2 ⇒ 3 ⇒ 4 ⇔ 5 ⇔ 6 ⇔ 7 hold.

Proof. Assuming 1, by Lemma 9, for all not necessarily simple graphs G and H, if $G \equiv_{\mathcal{F}} H$ and $\mathcal{F}$ is closed under deleting edges it holds that $\hom(F, \widehat{G}) = \hom(F, \widehat{H})$ for all $F \in \mathcal{F}$ as all graphs F' appearing in Lemma 9 are obtained from F by deleting edges. Hence, 2 holds. Clearly, 2 implies 3.

Suppose 3 holds. Since Lemma 6 is not directly applicable since the condition in 3 involves graphs with loops, we first prove the following claim:

Claim 13.1. *Let $\mathcal{F}$ be a graph class satisfying 3. For all graphs G and H with loops at every vertex, if $G \equiv_{\mathcal{F}} H$, then $G \equiv_{\operatorname{cl}(\mathcal{F})} H$.*

Proof of Claim. Let K be an arbitrary graph with a loop at every vertex. Then $G \times K$ and $H \times K$ are graph with loops at every vertex. Hence, their full complements $\widehat{G \times K}$ and $\widehat{H \times K}$ are simple graphs.

Let $F \in \operatorname{cl}(\mathcal{F})$. By Eq. (2), if $G \equiv_{\mathcal{F}} H$ then $G \times K \equiv_{\mathcal{F}} H \times K$. By 3, $\widehat{G \times K} \equiv_{\mathcal{F}} \widehat{H \times K}$, which implies that $\hom(F, \widehat{G \times K}) = \hom(F, \widehat{H \times K})$ since $\equiv_{\mathcal{F}}$ and $\equiv_{\operatorname{cl}(\mathcal{F})}$ coincide on simple graphs. By Lemma 9 and Eq. (2), this in turn implies that

$$\sum_{\substack{F' \subseteq F \\ V(F') = V(F)}} (-1)^{|E(F')|} \hom(F', G) \hom(F', K) = \sum_{\substack{F' \subseteq F \\ V(F') = V(F)}} (-1)^{|E(F')|} \hom(F', H) \hom(F', K). \quad (6)$$

Let $n \coloneqq |V(F)|$ and write $\mathcal{L}$ for the set of all simple graphs on at most n vertices. We claim that the matrix $(\hom(F, G^\circ)))_{F,G \in \mathcal{L}}$ is invertible. Indeed, by Theorem 10,

$$\hom(F, G^\circ) = \sum_{L \subseteq E(F)} \hom(F \oslash L, G) = \sum_{F' \in \mathcal{L}} |\{L \subseteq E(F) \mid F \oslash L \cong F'\}| \hom(F', G).$$

By classical arguments [1], the matrix $(\hom(F, G))_{F,G \in \mathcal{L}}$ is invertible. When ordering the elements of $\mathcal{L}$ first by number of vertices and then by number of edges, the matrix $(|\{L \subseteq E(F) \mid F \oslash L \cong F'\}|)_{F,F' \in \mathcal{L}}$ is upper triangular and all its diagonal entries are 1 since $F \oslash L \cong F$ iff $L = \emptyset$. Hence, $(\hom(F, G^\circ)))_{F,G\in\mathcal{L}}$ is invertible as the product of two invertible matrices. As in the proof of Lemma 6, one may multiply Eq. (6) with the inverse of this matrix to conclude that $\hom(F, G) = \hom(F, H)$. ⌟

By 3, for all simple graphs G and H, the assumption $G \equiv_{\mathcal{F}} H$ implies that $\widehat{G} \equiv_{\mathcal{F}} \widehat{H}$. Hence, by Claim 13.1, $\hom(F, \widehat{G}) = \hom(F, \widehat{H})$ for all $F \in \mathrm{cl}(\mathcal{F})$. Finally, by Lemma 9,

$$\sum_{\substack{F' \subseteq F \\ V(F')=V(F)}} (-1)^{|E(F')|} \hom(F', G) = \sum_{\substack{F' \subseteq F \\ V(F')=V(F)}} (-1)^{|E(F')|} \hom(F', H).$$

Since G and H are simple, we are in the setting of Lemma 6. Let K be any graph obtained from F by deleting edges, i.e. $V(K) = V(F)$ and $E(K) \subseteq E(F)$. Each summation index F' in Lemma 9 such that $F' \cong K$ contributes $(-1)^{|E(F')|} = (-1)^{|E(K)|}$. Hence, the coefficient of $\hom(K, -)$ in the linear combination obtained from the one above by grouping isomorphic summation indices is non-zero. By Lemma 6, it holds that $F' \in \mathrm{cl}(\mathcal{F})$.

The implication 4 ⇒ 5 follows from Lemma 12. The implication 5 ⇒ 6 follows from 1 ⇒ 3. The implication 6 ⇒ 7 is immediate. The final implication 7 ⇒ 4 follows from 3 ⇒ 4 observing that $\mathrm{cl}(\mathrm{cl}(\mathcal{F})) = \mathrm{cl}(\mathcal{F})$. □

3.4. Taking Induced Subgraphs, Contracting Edges, and Lexicographic Products

In this section, it is shown that a homomorphism distinguishing closed graph class is closed under taking induced subgraphs (contracting edges) if and only if its homomorphism indistinguishability relation is preserved under lexicographic products with a fixed graph from the left (from the right).

Examples for equivalence relations preserved under lexicographic products related to chromatic graph parameters are listed in Corollary 20. Further examples are of model-theoretic nature. It is for example easy to see that winning strategies of the Duplicator player in Hella's bijective pebble game [25] can be composed along lexicographic products.

A correspondence between taking induced subgraphs and left lexicographic products is established by the following Theorem 14:

Theorem 14. *For a graph class $\mathcal{F}$ and the assertions*

1. *$\mathcal{F}$ is closed under taking induced subgraphs,*
2. *$\equiv_{\mathcal{F}}$ is* preserved under left lexicographic products, *i.e. for all simple graphs G, H, and H', if $H \equiv_{\mathcal{F}} H'$ then $G \cdot H \equiv_{\mathcal{F}} G \cdot H'$,*
3. *$\mathrm{cl}(\mathcal{F})$ is closed under taking induced subgraphs.*

the implications 1 ⇒ 2 ⇔ 3 hold.

An example for a lexicographic product from the right is the *n-blow-up* $G \cdot \overline{K_n}$ of a graph G. It is well-known [22] that every homomorphism indistinguishability relation is preserved under blow-ups. Preservation under arbitrary lexicographic products from the right, however, is a non-trivial property corresponding to the associated graph class being closed under edge contractions:

Theorem 15. *For a graph class $\mathcal{F}$ and the assertions*

1. *$\mathcal{F}$ is closed under edge contractions,*
2. *$\equiv_{\mathcal{F}}$ is* preserved under right lexicographic products, *i.e. for all simple graphs G, G', and H, if $G \equiv_{\mathcal{F}} G'$ then $G \cdot H \equiv_{\mathcal{F}} G' \cdot H$,*
3. *$\mathrm{cl}(\mathcal{F})$ is closed under edge contractions.*

the implications 1 ⇒ 2 ⇔ 3 hold.

For the proofs of Theorems 14 and 15, homomorphism counts $\hom(F, G \cdot H)$ are written as linear combinations of homomorphism counts $\hom(F', H)$ where F' ranges over induced subgraphs of F in the case of Theorem 14 and over graphs obtained from F' by contracting edges in the case of Theorem 15. The following succinct formula for counts of homomorphisms into a lexicographic product may be of independent interest.

Theorem 16. *Let F, G, and H be simple graphs. Then*

$$\hom(F, G \cdot H) = \sum_{\mathcal{R}} \hom(F/\mathcal{R}, G) \hom(\coprod_{R \in \mathcal{R}} F[R], H)$$

where the outer sum ranges over all $\mathcal{R} \in \Pi(V(F))$ such that $F[R]$ is connected for all $R \in \mathcal{R}$.

Proof. The proof is by constructing a bijection between the set of homomorphisms $F \to G \cdot H$ and the set of triples $(\mathcal{R}, g, h)$ where $\mathcal{R} \in \Pi(V(F))$ is such that all $F[R]$, $R \in \mathcal{R}$, are connected, and $g\colon F/\mathcal{R} \to G$ and $h\colon \coprod_{R \in \mathcal{R}} F[R] \to H$ are homomorphisms. To that end, write $\pi_G\colon V(G \cdot H) \to V(G)$ and $\pi_H\colon V(G \cdot H) \to V(H)$ for the projection maps.

Let $f\colon F \to G \cdot H$ be a homomorphism. Define a partition $\mathcal{R}'$ of $V(F)$ with classes $(\pi_G \circ f)^{-1}(v)$ for $v \in V(G)$ and $\mathcal{R} \leq \mathcal{R}'$ as the coarsest partition whose classes $R \in \mathcal{R}$ all induce connected subgraphs $F[R]$, i.e. for every $R' \in \mathcal{R}'$, the partition $\mathcal{R}$ contains one class for every connected component of $F[R']$.

The homomorphism $g\colon F/\mathcal{R} \to G$ is given by the map sending $R \in \mathcal{R}$ to $(\pi_G \circ f)(v)$ for any $v \in R$. By definition of $\mathcal{R}$, this map is well-defined. It is indeed a homomorphism since if $R_1, R_2 \in \mathcal{R}$ are adjacent in $F/\mathcal{R}$, there exist $x_1 \in R_1$ and $x_2 \in R_2$ such that $x_1 x_2 \in E(F)$. Observe that $R_1 \neq R_2$ because $F/\mathcal{R}$ is simple and hence x_1 and x_2 lie in different classes of $\mathcal{R}'$. As $(\pi_G \circ f)(x_1) \neq (\pi_G \circ f)(x_2)$, it holds that $(\pi_G \circ f)(x_1)$ and $(\pi_G \circ f)(x_2)$ are adjacent vertices in G because f is a homomorphism. Hence, $g(R_1)$ and $g(R_2)$ are adjacent in G.

The homomorphism $h\colon \coprod_{R \in \mathcal{R}} F[R] \to H$ is given by $\pi_H \circ f$. This is indeed a homomorphism since if $x_1, x_2 \in V(F)$ are adjacent in $\coprod_{R \in \mathcal{R}} F[R]$ then they lie in the same class of $\mathcal{R}'$ and hence $h(x_1)h(x_2)$ is an edge of H.

For injectivity, suppose that $f, f'\colon F \to G \cdot H$ are both mapped to $(\mathcal{R}, g, h)$. Then, $\pi_H \circ f = h = \pi_H \circ f'$ and furthermore for every $v \in V(G)$ with $v \in R$ for some $R \in \mathcal{R}$, $(\pi_G \circ f)(v) = g(R) = g'(R) = (\pi_G \circ f)(v)$. Hence, $f = f'$.

For surjectivity, let $(\mathcal{R}, g, h)$ be a triple where $\mathcal{R} \in \Pi(V(F))$ is such that all $F[R]$, $R \in \mathcal{R}$, are connected, and $g\colon F/\mathcal{R} \to G$ and $h\colon \coprod_{R \in \mathcal{R}} F[R] \to H$ are homomorphisms. Define $f\colon F \to G \cdot H$ by $v \mapsto ((g \circ \rho)(v), h(v))$ where $\rho\colon F \to F/\mathcal{R}$ is the map sending $v \in V(F)$ to $R \in \mathcal{R}$ such that $v \in R$.

The map f is a homomorphism. Indeed, for $uv \in E(F)$, distinguish cases: If $(g \circ \rho)(u) = (g \circ \rho)(v)$ then $\rho(u) = \rho(v)$ since otherwise $\rho(u)$ and $\rho(v)$ are adjacent in $F/\mathcal{R}$ and $(g \circ \rho)(u) \neq (g \circ \rho)(v)$ as g is a homomorphism into a loop-less graph. In this case, $u, v \in R$ for some $R \in \mathcal{R}$ and thus $h(u)h(v)$ is an edge of H. If $(g \circ \rho)(u) \neq (g \circ \rho)(v)$ then in particular $\rho(u) \neq \rho(v)$, $\rho(u)\rho(v)$ is an edge of $F/\mathcal{R}$ and hence $(g \circ \rho)(u)$ and $(g \circ \rho)(v)$ are adjacent in G. In any case, $f(u)f(v)$ is an edge of $G \cdot H$.

Write $(\mathcal{R}', g', h')$ for the image of f under the aforementioned construction. It is claimed that $(\mathcal{R}', g', h') = (\mathcal{R}, g, h)$.

To argue that $\mathcal{R} = \mathcal{R}'$, let $x_1 \dots x_\ell$ be a path in F such that $(\pi_G \circ f)(x_1) = \dots = (\pi_G \circ f)(x_\ell)$. Then $(g \circ \rho)(x_1) = \dots = (g \circ \rho)(x_\ell)$ by construction. It has to be shown that $\rho(x_1) = \dots = \rho(x_\ell)$. If $\rho(x_i) \neq \rho(x_{i+1})$ for some $1 \leq i < \ell$ then $\rho(x_i)$ and $\rho(x_{i+1})$ are adjacent in $F/\mathcal{R}$, which cannot be since G is loop-less and both of these vertices have the same image under g. This implies that if x, y are in the same class of $\mathcal{R}'$ then they are in the same class of $\mathcal{R}$. Conversely, let $x_1 \dots x_\ell$ be a path in F such that all $x_1, \dots, x_\ell \in R$ for some $R \in \mathcal{R}$. Then $\rho(x_1) = \dots = \rho(x_\ell)$ and hence $(g \circ \rho)(x_1) = \dots = (g \circ \rho)(x_\ell)$. In particular, if x, y are in the same class of $\mathcal{R}$ then they are in the same class under $\mathcal{R}'$.

To argue that $g = g'$, note that for any $R \in \mathcal{R} = \mathcal{R}'$ with $v \in R$, $g'(R) = (\pi_G \circ f)(v) = (g \circ \rho)(v) = g(R)$. Finally, $h' = \pi_H \circ f = h$. $\square$

Theorem 16 yields Theorems 14 and 15 as follows:

Proof of Theorem 14. That 1 implies 2 is immediate from Theorem 16.

Let $F \in \mathrm{cl}(\mathcal{F})$ and $U \subseteq V(F)$. In order to show that $F[U] \in \mathrm{cl}(\mathcal{F})$ assuming 2, let $G = K_n$ where $n \coloneqq |V(F)|$. Observe that $\hom(F/\mathcal{P}, G) > 0$ for every $\mathcal{P} \in \Pi(V(F))$. For the discrete partition $\mathcal{D}$, $\coprod_{R \in \mathcal{D}} F[R] \cong nK_1$. By Lemma 6, $K_1 \in \mathrm{cl}(\mathcal{F})$. Write $\mathcal{P}$ for the partition whose classes are formed by the connected components of $F[U]$ and singleton classes otherwise. Then $\coprod_{P \in \mathcal{P}} F[P] \cong F[U] + mK_1$ where $m \coloneqq |V(F) \setminus U|$. By Lemma 6, $F[U] + mK_1 \in \mathrm{cl}(\mathcal{F})$ which implies, as for example argued in Lemma 12, that $F[U] \in \mathrm{cl}(\mathcal{F})$. Hence, 2 implies 3.

The implication $3 \Rightarrow 2$ follows from $1 \Rightarrow 2$ for $\mathrm{cl}(\mathcal{F})$ since $\equiv_{\mathcal{F}}$ and $\equiv_{\mathrm{cl}(\mathcal{F})}$ coincide. □

Proof of Theorem 15. That 1 implies 2 is immediate from Theorem 16.

Let $F \in \mathrm{cl}(\mathcal{F})$. It is to show that $F/\mathcal{P} \in \mathrm{cl}(\mathcal{F})$ for every $\mathcal{P} \in \Pi(V(F))$ such that all $F[P]$ are connected for $P \in \mathcal{P}$. To that end, write $n \coloneqq |V(F)|$ and let $H = K_n$. Observe that $\hom(F[U], H) > 0$ for every $U \subseteq V(F)$. Hence, the coefficient of $\hom(F/\mathcal{P}, -)$ in the linear combination from Theorem 16 is non-zero. By Lemma 6, $F/\mathcal{P} \in \mathrm{cl}(\mathcal{F})$. Hence, 2 implies 3.

The implication $3 \Rightarrow 2$ follows from $1 \Rightarrow 2$ for $\mathrm{cl}(\mathcal{F})$ since $\equiv_{\mathcal{F}}$ and $\equiv_{\mathrm{cl}(\mathcal{F})}$ coincide. □

Theorems 14 and 15 can be combined to yield the following Corollary 17:

Corollary 17. *For a graph class $\mathcal{F}$ and the assertions*

1. *$\mathcal{F}$ is closed under taking induced subgraphs and edge contractions,*
2. *$\equiv_{\mathcal{F}}$ is such that for all simple graphs G, G', H, and H', if $G \equiv_{\mathcal{F}} G'$ and $H \equiv_{\mathcal{F}} H'$ then $G \cdot H \equiv_{\mathcal{F}} G' \cdot H'$,*
3. *$\mathrm{cl}(\mathcal{F})$ is closed under taking induced subgraphs and edge contractions.*

the implications $1 \Rightarrow 2 \Leftrightarrow 3$ hold.

Proof. Assuming 1, if $G \equiv_{\mathcal{F}} G'$ and $H \equiv_{\mathcal{F}} H'$ then $G \cdot H \equiv_{\mathcal{F}} G' \cdot H$ by Theorem 15 and $G' \cdot H \equiv_{\mathcal{F}} G' \cdot H'$ by Theorem 14. By transitivity, $G \cdot H \equiv_{\mathcal{F}} G' \cdot H'$ and 2 holds. The implication $2 \Rightarrow 3$ is immediate from Theorems 14 and 15. The implication $3 \Rightarrow 2$ follows from $1 \Rightarrow 2$ for $\mathrm{cl}(\mathcal{F})$ since $\equiv_{\mathcal{F}}$ and $\equiv_{\mathrm{cl}(\mathcal{F})}$ coincide. □

As a final observation, the following Lemma 18 relates the property of being closed under edge contractions to the other closure properties in Table 1 and Fig. 1.

Lemma 18. *If a homomorphism distinguishing closed graph class $\mathcal{F}$ is closed under contracting edges then it is closed under taking summands.*

Proof. Let $F \in \mathcal{F}$. Since every homomorphism distinguishing closed graph class is closed under disjoint unions, cf. Eq. (1) and [10], it suffices to show that every connected component C of F is in $\mathcal{F}$. Let m denote the number of connected components of F. By contracting all edges, the graph mK_1 can be obtained from F. Hence, as argued in Lemma 12, $K_1 \in \mathcal{F}$. Moreover, the graph $C + (m-1)K_1$ can be obtained from F by contracting all edges not in C. This implies as in Lemma 12 that $C \in \mathcal{F}$. □

3.5. Applications

As applications of Theorems 8 and 13 to 15, we conclude, in the spirit of [26], that certain equivalence relations on graphs cannot be homomorphism distinguishing relations.

Corollary 19. *Let $\mathcal{F}$ be a graph class containing a non-empty graph such that one of the following holds:*

1. *$\equiv_{\mathcal{F}}$ is preserved under complements, cf. Theorem 8,*
2. *$\equiv_{\mathcal{F}}$ is preserved under full complements, cf. Theorem 13,*
3. *$\equiv_{\mathcal{F}}$ is preserved under left lexicographic products, cf. Theorem 14, or*
4. *$\equiv_{\mathcal{F}}$ is preserved under right lexicographic products, cf. Theorem 15.*

Then $G \equiv_{\mathcal{F}} H$ implies that $|V(G)| = |V(H)|$ for all graphs G and H.

Proof. By Theorems 8 and 13 to 15, $\mathcal{F}$ can be chosen to be closed under taking minors, subgraphs, induced subgraphs, or contracting edges. In any case, $K_1 \in \mathcal{F}$ and hence $|V(G)| = \hom(K_1, G) = \hom(K_1, H) = |V(H)|$. ☐

As concrete examples, consider the following relations.

Corollary 20. *There is no graph class $\mathcal{F}$ satisfying any of the following assertions for all graphs G and H:*

1. $G \equiv_{\mathcal{F}} H$ *iff* $a(G) = a(H)$ *where a denotes the order of the automorphism group,*
2. $G \equiv_{\mathcal{F}} H$ *iff* $\alpha(G) = \alpha(H)$ *where α denotes the size of the largest independent set,*
3. $G \equiv_{\mathcal{F}} H$ *iff* $\omega(G) = \omega(H)$ *where ω denotes the size of the largest clique,*
4. $G \equiv_{\mathcal{F}} H$ *iff* $\chi(G) = \chi(H)$ *where χ denotes the chromatic number.*

Proof. The relation in Item 1 is preserved under taking complements. By [27, Theorem 1, Corollary p. 90], the relations in Items 2 and 4 are preserved under left lexicographic products. For Item 3, the same follows from [27, Theorem 1] observing that $\overline{G} \cdot \overline{H} = \overline{G \cdot H}$ and $\omega(G) = \alpha(\overline{G})$. In each case, it is easy to exhibit a pair of graphs G and H in the same equivalence class with different number of vertices. By Corollary 19, none of the equivalence relations is a homomorphism indistinguishability relation. ☐

4. Equivalences over Self-Complementary Logics

In this section, Theorem 1 is derived from Theorem 8. The theorem applies to self-complementary logics, of which examples are given subsequently. Finally, a result from graph minor theory is used to relate logics on graphs to quantum isomorphism.

A *logic on graphs* [9] is a pair $(\mathsf{L}, \models)$ of a class L and a relation $\models$ comparing simple graphs and elements of L which is isomorphism-invariant, i.e. satisfying that for all $\varphi \in \mathsf{L}$ and simple graphs G and H such that $G \cong H$ it holds that $G \models \varphi$ iff $H \models \varphi$. When convenient, we omit the reference to $\models$ and denote $(\mathsf{L}, \models)$ by L. Two simple graphs G and H are L-*equivalent* if for all $\varphi \in \mathsf{L}$ it holds that $G \models \varphi$ iff $H \models \varphi$. One may think of a logic on graphs as a mere collection of isomorphism-invariant graph properties. Every $\varphi \in \mathsf{L}$ defines such property. This very general definition has to be strengthened only slightly in order to yield Theorem 2:

Definition 21. A logic on graphs $(\mathsf{L}, \models)$ is *self-complementary* if for all $\varphi \in \mathsf{L}$ there is a $\overline{\varphi} \in \mathsf{L}$ such that for all simple graphs G it holds that $G \models \varphi$ if and only if $\overline{G} \models \overline{\varphi}$.

Theorem 1. *Let $(\mathsf{L}, \models)$ be a self-complementary logic on graphs for which there exists a graph class $\mathcal{F}$ such that simple graphs are homomorphism indistinguishable over $\mathcal{F}$ if and only if they are L-equivalent. Then there exists a minor-closed graph class $\mathcal{F}'$ whose homomorphism indistinguishability relation coincides with L-equivalence.*

Proof. It is shown that $\equiv_{\mathcal{F}}$ is preserved under taking complements in the sense of Theorem 8. Let G and H be simple graphs such that $G \equiv_{\mathcal{F}} H$. By assumption, for all $\varphi \in \mathsf{L}$ it holds that $G \models \varphi$ iff $H \models \varphi$ and hence, by self-complementarity,

$$\overline{G} \models \varphi \iff G \models \overline{\varphi} \iff H \models \overline{\varphi} \iff \overline{H} \models \overline{\overline{\varphi}} \iff \overline{H} \models \varphi.$$

Here, the penultimate equivalence holds since $H \models \varphi$ if and only if $H \models \overline{\overline{\varphi}}$ for all φ and H by the definition of self-complementarity observing that $\overline{\overline{H}} \cong H$. Thus, $\overline{G} \equiv_{\mathcal{F}} \overline{H}$. By Theorem 8, $\mathcal{F}' := \mathrm{cl}(\mathcal{F})$ is minor-closed. ☐

In particular, by Corollary 19, for a self-complementary logic L, all L-equivalent simple graphs G and H must have the same number of vertices unless L is trivial in the sense that all simple graphs G and H are L-equivalent.

4.1. Examples for Self-Complementary Logics

A first example of a self-complementarity logic is first-order logic FO over the signature of graphs $\{E\}$. In order to establish this property, a formula $\overline{\varphi} \in \mathsf{FO}$ has to be constructed for every $\varphi \in \mathsf{FO}$ such that $G \models \varphi$ iff $\overline{G} \models \overline{\varphi}$ for all simple graphs G. Only subformulae Exy require non-trivial treatment:

Definition 22. For every $\varphi \in \mathsf{FO}$, define $\overline{\varphi} \in \mathsf{FO}$ inductively as follows:

1. if $\varphi = Exy$ then $\overline{\varphi} := \neg Exy \wedge (x \neq y)$,
2. if $\varphi = \bot$ or $\varphi = \top$ then $\overline{\varphi} := \varphi$, if $\varphi = (x = y)$ then $\overline{\varphi} := \varphi$, if $\varphi = \neg\psi$ then $\overline{\varphi} := \neg\overline{\psi}$, if $\varphi = \psi \wedge \chi$ then $\overline{\varphi} := \overline{\psi} \wedge \overline{\chi}$, if $\varphi = \psi \vee \chi$ then $\overline{\varphi} := \overline{\psi} \vee \overline{\chi}$, if $\varphi = \exists x.\ \psi$ then $\overline{\varphi} := \exists x.\ \overline{\psi}$, and if $\varphi = \forall x.\ \psi$ then $\overline{\varphi} := \forall x.\ \overline{\psi}$.

Lemma 23. *Let $\varphi \in \mathsf{FO}$ be a formula with $k \geq 0$ free variables. Then for all simple graphs G with $\boldsymbol{v} \in V(G)^k$ it holds that*

$$G, \boldsymbol{v} \models \varphi \iff \overline{G}, \boldsymbol{v} \models \overline{\varphi}.$$

In particular, FO is self-complementary.

Proof. The proof is by induction on the structure of φ. If φ is Exy, observe that $v_1v_2 \in E(G)$ if and only if $v_1v_2 \notin E(\overline{G})$ and $v_1 \neq v_2$. In all other cases, the claim is purely syntactical. $\square$

Lemma 23 gives a purely syntactical criterion for a fragment $\mathsf{L} \subseteq \mathsf{FO}$ to be self-complementary. Indeed, if $\overline{\varphi} \in \mathsf{L}$ as defined in Theorem 22 for all $\varphi \in \mathsf{L}$ then L is self-complementary. Note that the operation in Theorem 22 increases neither the number of variables nor affects the quantifiers in the formula. Thus, Lemma 23 automatically extends to fragments of FO defined by restricting the number of variables, order or number of quantifiers. For extensions of FO, Theorem 22 can be easily extended. This yields a rich realm of self-complementary logics, of which the following Example 24 lists only a selection.

Example 24. *The following logics on graphs are self-complementary. For every $k, d \geq 0$,*

1. *the k-variable and quantifier-depth-d fragments FO^k and FO_d of FO,*
2. *first-order logic with counting quantifiers C and its k-variable and quantifier-depth-d fragments C^k and C_d,*
3. *inflationary fixed-point logic IFP, cf. [28],*
4. *second-order logic SO and its fragments monadic second-order logic MSO_1, existential second-order logic ESO, cf. [29, 30].*

Corollary 19 readily gives an alternative proof of [26, Propositions 1 and 2], which assert that neither FO^k-equivalence nor FO_d-equivalence are characterised by homomorphism indistinguishability relations. The logic fragments C^k and C_d are however characterised by homomorphism indistinguishability relations [8, 4].

4.2. Applications: Graph Minor Theory and Expressive Power

The final result of this section demonstrates how graph minor theory can yield insights into the expressive power of logics via Theorem 1. Subject to it are self-complementary logics which have a homomorphism indistinguishability characterisation and are stronger than C^k for every k, e.g. they are capable of distinguishing CFI-graphs [31]. It is shown that equivalence w.r.t. any such logic is a sufficient condition for quantum isomorphism, an undecidable equivalence comparing graphs [32].

Theorem 25. *Let $(\mathsf{L}, \models)$ be a self-complementary logic on graphs for which there exists a graph class $\mathcal{F}$ such that simple graphs are homomorphism indistinguishable over $\mathcal{F}$ if and only if they are L-equivalent. Suppose that for all $k \in \mathbb{N}$ there exist simple graphs G and H such that $G \equiv_{\mathsf{C}^k} H$ and $G \not\equiv_{\mathsf{L}} H$. Then all L-equivalent graphs are quantum isomorphic.*

Proof. Contrapositively, it is shown that if there exist non-quantum-isomorphic L-equivalent graphs then there exists a $k \in \mathbb{N}$ such that $G \equiv_{\mathsf{C}^k} H \implies G \equiv_{\mathsf{L}} H$ for all simple graphs G and H. By [3, 8], this statement can be rephrased in the language of homomorphism indistinguishability as $\mathcal{P} \not\subseteq \mathrm{cl}(\mathcal{F}) \implies \exists k \in \mathbb{N}.\ \mathcal{F} \subseteq \mathrm{cl}(\mathcal{TW}_k)$ where $\mathcal{P}$ denotes the class of all planar graphs and $\mathcal{TW}_k$ the class of all graphs of treewidth at most k. By Theorem 8, $\mathrm{cl}(\mathcal{F})$ is a minor-closed graph class. By [33, (2.1)], cf. [34, Theorem 3.8], if $\mathrm{cl}(\mathcal{F})$ does not contain all planar graphs then it is of bounded treewidth. Hence, there exists a $k \in \mathbb{N}$ such that $\mathcal{F} \subseteq \mathrm{cl}(\mathcal{F}) \subseteq \mathcal{TW}_k \subseteq \mathrm{cl}(\mathcal{TW}_k)$. □

5. Classification of Homomorphism Distinguishing Closed Essentially Profinite Graph Classes

The central result of this section is a classification of the homomorphism distinguishing closed graph classes which are in a sense finite. Since every homomorphism distinguishing closed graph class is closed under disjoint unions, infinite graph classes arise naturally when studying the semantic properties of the homomorphism indistinguishability relations of finite graph classes. Nevertheless, the infinite graph classes arising in this way are *essentially finite*, i.e. they exhibit only finitely many distinct connected components. One may generalise this definition slightly by observing that all graphs F admitting a homomorphism into some fixed graph G have chromatic number bounded by the chromatic number of G. Thus, in order to make a graph class $\mathcal{F}$ behave much like an essentially finite class, it suffices to impose a finiteness condition, for every graph K, on the subfamily of all K-colourable graphs in $\mathcal{F}$.

Formally, for a graph F, write $\Gamma(F)$ for the set of connected components of F. For a graph class $\mathcal{F}$, define $\Gamma(\mathcal{F})$ as the union of the $\Gamma(F)$ where $F \in \mathcal{F}$. For a graph class $\mathcal{F}$ and a graph K, define $\mathcal{F}_K := \{F \in \mathcal{F} \mid \hom(F, K) > 0\}$, the set of K-colourable graphs in $\mathcal{F}$.

Definition 26. A graph class $\mathcal{F}$ is *essentially finite* if $\Gamma(\mathcal{F})$ is finite. It is *essentially profinite* if $\mathcal{F}_K$ is essentially finite for every simple graph K.

Clearly, every finite graph class is essentially finite and hence essentially profinite. Other examples for essentially profinite classes are the class of all cliques. They represent a special case of the following construction from [10, Theorem 6.16]: For every $S \subseteq \mathbb{N}$, the family

$$\mathcal{K}^S := \{K_{n_1} + \dots + K_{n_r} \mid r \in \mathbb{N}, \{n_1, \dots, n_r\} \subseteq S\} \tag{7}$$

is essentially profinite. In particular, there are uncountably many such families of graphs. Note that one may replace the sequence of cliques $(K_n)_{n\in\mathbb{N}}$ in Eq. (7) by any other sequence of connected graphs $(F_n)_{n\in\mathbb{N}}$ such that the sequence of chromatic numbers $(\chi(F_n))_{n\in\mathbb{N}}$ takes every value only finitely often.

Every graph F of an essentially finite family $\mathcal{F}$ can be represented uniquely as vector $\vec{F} \in \mathbb{R}^{\Gamma(\mathcal{F})}$ whose C-th entry for $C \in \Gamma(\mathcal{F})$ is the number of times the graph C appears as a connected component in F. The classification of the homomorphism distinguishing closed essentially profinite graph classes can now be stated as follows.

Theorem 27. *For an essentially profinite graph class $\mathcal{F}$, the following are equivalent:*

1. *$\mathcal{F}$ is homomorphism distinguishing closed,*
2. *For every simple graph K, if $\vec{K} \in \mathrm{span}\{\vec{F} \in \mathbb{R}^{\Gamma(\mathcal{F}_K \cup \{K\})} \mid F \in \mathcal{F}_K\}$ then $K \in \mathcal{F}$,*
3. *$\mathcal{F}_K$ is homomorphism distinguishing closed for every simple graph K.*

Theorem 27 directly implies Theorem 3. Indeed, if $\mathcal{F}$ is union-closed and closed under summands then $\Gamma(\mathcal{F}) \subseteq \mathcal{F}$ and every graph K such that $\Gamma(K) \subseteq \Gamma(\mathcal{F})$ is itself in $\mathcal{F}$. In particular, Theorem 27 implies that all essentially profinite union-closed minor-closed graph classes are homomorphism distinguishing closed. For example, for every graph G, the union-closure of the class of minors of G is homomorphism distinguishing closed, cf. [10, Question 4].

Towards proving Theorem 27, we first make the following general observation: Considering essentially profinite graph classes is very natural in light of the following Lemma 28. For every graph K, the subset $\mathcal{F}_K$ of $\mathcal{F}$ is the object prescribing whether $K \in \mathrm{cl}(\mathcal{F})$.

Lemma 28. *Let $\mathcal{F}$ be a graph class and K be a simple graph. Then $K \in \mathrm{cl}(\mathcal{F})$ if and only if $K \in \mathrm{cl}(\mathcal{F}_K)$.*

Proof. The backward implication is immediate since $\mathcal{F}_K \subseteq \mathcal{F}$ and thus $\mathrm{cl}(\mathcal{F}_K) \subseteq \mathrm{cl}(\mathcal{F})$. Conversely, suppose that $K \in \mathrm{cl}(\mathcal{F})$. Let G and H be arbitrary graphs such that $G \equiv_{\mathcal{F}_K} H$. Then, by Eq. (2), $G \times K \equiv_{\mathcal{F}} H \times K$ since $\hom(F, K) = 0$ for all $F \in \mathcal{F} \setminus \mathcal{F}_K$. By assumption, $\hom(K, G \times K) = \hom(K, H \times K)$, and therefore $\hom(K, G) = \hom(K, H)$ since $\hom(K, K) > 0$. Thus, $K \in \mathrm{cl}(\mathcal{F}_K)$. □

The proof of Theorem 27 is based on a generalisation of a result by Kwiecień, Marcinkowski, and Ostropolski-Nalewaja [20]. They proved the following Theorem 29 for finite graph classes. The extension to essentially finite graph classes does not require much additional work but might make core ideas appear more transparently.

Theorem 29 (Kwiecień–Marcinkowski–Ostropolski-Nalewaja [20])**.** *Let $\mathcal{F}$ be an essentially finite family of graphs and K be a simple graph. Suppose that $\hom(F, K) > 0$ for all $F \in \mathcal{F}$. Let $\mathcal{C} \coloneqq \Gamma(\mathcal{F} \cup \{K\})$. Then $K \in \mathrm{cl}(\mathcal{F})$ if and only if $\vec{K} \in \mathrm{span}\{\vec{F} \in \mathbb{R}^{\mathcal{C}} \mid F \in \mathcal{F}\}$.*

Proof. For the backward direction, suppose that $\vec{K} \in \mathrm{span}\{\vec{F} \in \mathbb{R}^{\mathcal{C}} \mid F \in \mathcal{F}\}$. Observe that this implies that $\Gamma(\mathcal{F}) = \mathcal{C}$, i.e. all connected components of K appear as connected components of some $F \in \mathcal{F}$. Write $\vec{K} = \sum_{i=1}^{r} \alpha_i \vec{F}_i$ for some $\alpha_1, \dots, \alpha_r \in \mathbb{R}$ and $F_1, \dots, F_r \in \mathcal{F}$. Also write $K = \coprod_{C \in \mathcal{C}} \beta_C C$ for some $\beta_C \in \mathbb{N}$. Observe that $\beta_C = \vec{K}_C = \sum_{i=1}^{r} \alpha_i (\vec{F}_i)_C$ for all $C \in \mathcal{C}$.

Let G and H be graphs such that $G \equiv_{\mathcal{F}} H$. If $\hom(F, G) = 0 = \hom(F, H)$ for some $F \in \mathcal{F}$ then $\hom(K, G) = 0 = \hom(K, H)$ by the assumption that $\hom(F, K) > 0$ for all $F \in \mathcal{F}$. Hence, it may be supposed that $\hom(F, G) = \hom(F, H) > 0$ for all $F \in \mathcal{F}$. This is crucial for ruling out division by zero in the following argument. Observe that

$$\hom(F, -) = \prod_{C \in \mathcal{C} \text{ s.t. } \vec{F}_C \geq 1} \hom(C, -)^{\vec{F}_C}$$

for all $F \in \mathcal{F} \cup \{K\}$. It follows that $\hom(C, G) > 0$ and $\hom(C, H) > 0$ for all $C \in \Gamma(\mathcal{F}) = \mathcal{C}$. Thus, in every step of the following calculation, the bases of all exponentials are positive integers.

$$\begin{aligned}\hom(K, G) &= \prod_{C \in \mathcal{C}} \hom(C, G)^{\beta_C} = \prod_{C \in \mathcal{C}} \hom(C, G)^{\sum_{i=1}^{r} \alpha_i (\vec{F}_i)_C} = \prod_{i=1}^{r} \prod_{C \in \mathcal{C}} \hom(C, G)^{\alpha_i (\vec{F}_i)_C} \\ &= \prod_{i=1}^{r} \hom(F_i, G)^{\alpha_i} = \prod_{i=1}^{r} \hom(F_i, H)^{\alpha_i} = \hom(K, H).\end{aligned}$$

Conversely, pick a finite family of graphs $\mathcal{G}$ such that the matrix $M \coloneqq (\hom)|_{\mathcal{C} \times \mathcal{G}}$ is invertible, e.g. in virtue of [22, Proposition 5.44(b)]. Consider the map

$$\begin{aligned}\Psi \colon \mathbb{R}^{\mathcal{G}} &\to \mathbb{R}^{\mathcal{C}} \\ a &\mapsto \sum_{G \in \mathcal{G}} \hom(\mathcal{C}, G) a_G = Ma.\end{aligned}$$

We think of $\mathbb{N}^{\mathcal{G}} \subseteq \mathbb{R}^{\mathcal{G}}$ as the space of instructions for constructing graphs as disjoint unions of elements in $\mathcal{G}$. The vector $a \in \mathbb{N}^{\mathcal{G}}$ corresponds to the graph $\coprod_{G \in \mathcal{G}} a_G G$. The map Ψ associates with such a graph its $\mathcal{C}$-homomorphism vector. In this way, $\mathbb{R}^{\mathcal{C}}$ may be thought of as the space of $\mathcal{C}$-homomorphism vectors. As a vector space isomorphism, Ψ is a homeomorphism.

Contrapositively, suppose that $\vec{K} \notin \mathrm{span}\{\vec{F} \in \mathbb{R}^{\mathcal{C}} \mid F \in \mathcal{F}\}$. Pick an integer vector $z \in \mathbb{Z}^{\mathcal{C}}$ such that $\langle z, \vec{F} \rangle = 0$ for all $F \in \mathcal{F}$ and $\langle z, \vec{K} \rangle \neq 0$. This can be done by Gram–Schmidt orthogonalisation applied to the rational vectors spanning $\mathrm{span}\{\vec{F} \in \mathbb{R}^{\mathcal{C}} \mid F \in \mathcal{F}\}$ and to the rational vector $\vec{K}$. The resulting vector z' with rational entries is orthogonal to all $\vec{F}$, $F \in \mathcal{F}$ and has non-zero inner-product with z. The integer vector z is then obtained from z' by clearing denominators.

Pick $p \in \Psi(\mathbb{Q}^{\mathcal{G}}_{>0}) \subseteq \mathbb{Q}^{\mathcal{C}}_{>0}$. In what follows, the vector p is perturbed in a direction depending on z. Technical complications arise when proving that the perturbed vector can again be interpreted as an instruction for constructing a graph, i.e. has a positive rational preimage under Ψ. To that end, consider the continuous function $\varphi\colon \mathbb{R}_{>0} \to \mathbb{R}^{\mathcal{C}}_{>0}$ which maps $t \mapsto (t^{z_C} p_C \mid C \in \mathcal{C})$.

Claim 29.1. *There exists $t > 1$ such that $\varphi(t) \in \Psi(\mathbb{Q}^{\mathcal{G}}_{>0})$.*

Proof of Claim. As the image of an open set under a homeomorphism, the set $\Psi(\mathbb{R}^{\mathcal{G}}_{>0})$ is open in $\mathbb{R}^{\mathcal{C}}$. Hence, there exists $\varepsilon > 0$ such that $B_\varepsilon(p) \subseteq \Psi(\mathbb{R}^{\mathcal{G}}_{>0})$. Since $\varphi^{-1}(B_\varepsilon(p))$ is open, $\varphi(1) = p$, and $\mathbb{Q}_{>0}$ is dense in $\mathbb{R}_{>0}$, there exists a rational $t > 1$ such that $\varphi(t) \in B_\varepsilon(p) \subseteq \Psi(\mathbb{R}^{\mathcal{G}}_{>0})$. Because t and p are rational and z is integral, $\varphi(t) \in \mathbb{Q}^{\mathcal{C}}$. The matrix M from definition of Ψ has integer entries and hence its preimages of rational vectors are rational. This implies that $\varphi(t) \in \Psi(\mathbb{Q}^{\mathcal{G}}_{>0})$. ⌟

Write $p' := \varphi(t)$ for the t whose existence is guaranteed by Claim 29.1. Let $s, s' \in \mathbb{Q}^{\mathcal{G}}_{>0}$ denote the preimages of p and p' under Ψ, respectively. There exists a natural number $\lambda \geq 1$ such that λs and $\lambda s'$ lie in $\mathbb{N}^{\mathcal{G}}$. Write H and H' for the structures obtained by interpreting λs and $\lambda s'$ as instructions for disjoint unions over elements in $\mathcal{G}$, i.e. $H = \coprod_{G \in \mathcal{G}} (\lambda s)_G G$ and $H' = \coprod_{G \in \mathcal{G}} (\lambda s')_G G$. Then

$$\hom(F, H) = \prod_{C \in \mathcal{C}} \hom(C, H)^{\vec{F}_C} = \prod_{C \in \mathcal{C}} \Psi(\lambda s)_C^{\vec{F}_C} = \prod_{C \in \mathcal{C}} (\lambda \Psi(s)_C)^{\vec{F}_C} = \prod_{C \in \mathcal{C}} (\lambda p_C)^{\vec{F}_C}$$

and, similarly,

$$\hom(F, H') = \prod_{C \in \mathcal{C}} \Psi(\lambda s')_C^{\vec{F}_C} = \prod_{C \in \mathcal{C}} (\lambda \Psi(s')_C)^{\vec{F}_C} = \prod_{C \in \mathcal{C}} (\lambda p'_C)^{\vec{F}_C} = t^{\langle z, \vec{F} \rangle} \prod_{C \in \mathcal{C}} (\lambda p_C)^{\vec{F}_C}$$

for all graphs F which are disjoint unions of graphs in $\mathcal{C}$. In particular, $\hom(F, H) = \hom(F, H')$ for all $F \in \mathcal{F}$ but $\hom(K, H) \neq \hom(K, H')$. Thus $K \notin \mathrm{cl}(\mathcal{F})$. □

Towards the proof of Theorem 27, we collect the following lemmas:

Lemma 30. *For every graph class $\mathcal{F}$ and every graph simple K, $\mathrm{cl}(\mathcal{F})_K \subseteq \mathrm{cl}(\mathcal{F}_K)$.*

Proof. Consider the following chain of inclusions:

$$\mathrm{cl}(\mathcal{F})_K = \bigcup_{L \in \mathrm{cl}(\mathcal{F})_K} \{L\} \subseteq \bigcup_{L \in \mathrm{cl}(\mathcal{F})_K} \mathrm{cl}(\mathcal{F}_L) \subseteq \mathrm{cl}\left(\bigcup_{L \in \mathrm{cl}(\mathcal{F})_K} \mathcal{F}_L\right) \subseteq \mathrm{cl}(\mathcal{F}_K).$$

The first inclusion follows from Lemma 28. Indeed, if $L \in \mathrm{cl}(\mathcal{F})$ then $L \in \mathrm{cl}(\mathcal{F}_L)$. The second inclusion is implied by Lemma 4. The third inclusion holds since if $F \in \mathcal{F}_L$ for $L \in \mathrm{cl}(\mathcal{F})_K$ then $\hom(F, K) > 0$. □

For essentially profinite graph classes, Theorem 29 implies that the converse of Lemma 30 holds.

Lemma 31. *For every essentially profinite $\mathcal{F}$, it holds that $\mathrm{cl}(\mathcal{F}_K) = \mathrm{cl}(\mathcal{F})_K$ for every simple graph K.*

Proof. That $\mathrm{cl}(\mathcal{F})_K \subseteq \mathrm{cl}(\mathcal{F}_K)$ is the assertion of Lemma 30. Conversely, $\mathrm{cl}(\mathcal{F}_K) \subseteq \mathrm{cl}(\mathcal{F})$ since $\mathcal{F}_K \subseteq \mathcal{F}$. It remains to argue that every $F \in \mathrm{cl}(\mathcal{F}_K)$ is K-colourable. If $F \in \mathrm{cl}(\mathcal{F}_K)$ then, by Theorem 29, $\Gamma(F) \subseteq \Gamma(\mathcal{F}_K)$. In other words, all connected components of F are K-colourable. This implies that F is K-colourable. Hence, $\mathrm{cl}(\mathcal{F}_K) \subseteq \mathrm{cl}(\mathcal{F})_K$. □

The concludes the preparations for the proof of Theorem 27.

Proof of Theorem 27. Suppose that $\mathcal{F}$ is homomorphism distinguishing closed. Let K be a graph such that $\Gamma(K) \subseteq \Gamma(\mathcal{F}_K)$. If $\vec{K} \in \operatorname{span}\{\vec{F} \in \mathbb{R}^{\Gamma(\mathcal{F}_K)} \mid F \in \mathcal{F}_K\}$ then Theorem 29 applies. Hence, $K \in \operatorname{cl}(\mathcal{F})$ and thus $K \in \mathcal{F}$ since $\mathcal{F}$ is homomorphism distinguishing closed.

Assuming Item 2, let $L \notin \mathcal{F}$. It follows that $\vec{L} \notin \operatorname{span}\{\vec{F} \in \mathbb{R}^{\Gamma(\mathcal{F}_L \cup \{L\})} \mid F \in \mathcal{F}_L\}$. Indeed, if $\Gamma(L) \subseteq \Gamma(\mathcal{F}_L)$ then this holds by Item 2. If $\Gamma(L) \not\subseteq \Gamma(\mathcal{F}_L)$ then L has a connected component which is not in $\mathcal{F}_L$ and the claim follows readily. Hence, by Theorem 29 and Lemma 28, $L \notin \operatorname{cl}(\mathcal{F})$. So $\mathcal{F}$ is homomorphism distinguishing closed.

The equivalence of Items 1 and 3 follows from Lemma 31. Indeed, if $\mathcal{F}$ is homomorphism distinguishing closed then $\operatorname{cl}(\mathcal{F}_K) = \operatorname{cl}(\mathcal{F})_K = \mathcal{F}_K$ for every K. Conversely,

$$\operatorname{cl}(\mathcal{F}) = \bigcup_K \operatorname{cl}(\mathcal{F})_K = \bigcup_K \operatorname{cl}(\mathcal{F}_K) = \bigcup_K \mathcal{F}_K = \mathcal{F}. \qquad \square$$

Finally, we deduce the following Corollaries 32 and 33 from Theorem 29:

Corollary 32. *If a graph class $\mathcal{F}$ is essentially finite then $\operatorname{cl}(\mathcal{F})$ is essentially finite.*

Proof. Let $K \in \operatorname{cl}(\mathcal{F})$. By Lemma 28, $K \in \operatorname{cl}(\mathcal{F}_K)$. By Theorem 29, $\vec{K} \in \operatorname{span}\{\vec{F} \in \mathbb{R}^{\Gamma(\mathcal{F}_K \cup \{K\})} \mid F \in \mathcal{F}_K\}$. In particular, $\Gamma(K) \subseteq \Gamma(\mathcal{F}_K)$. Hence, $\Gamma(\operatorname{cl}(\mathcal{F})) = \bigcup_{K \in \operatorname{cl}(\mathcal{F})} \Gamma(K) \subseteq \bigcup_{K \in \operatorname{cl}(\mathcal{F})} \Gamma(\mathcal{F}_K) \subseteq \Gamma(\mathcal{F})$. Thus, $\operatorname{cl}(\mathcal{F})$ is essentially finite. $\square$

Since the class of all graphs is not essentially profinite, the following Corollary 33 implies that no homomorphism indistinguishability relation of an essentially profinite graph class is as fine as isomorphism.

Corollary 33. *If a graph class $\mathcal{F}$ is essentially profinite then $\operatorname{cl}(\mathcal{F})$ is essentially profinite.*

Proof. It has to be argued that for every K the class $\operatorname{cl}(\mathcal{F})_K$ is essentially finite. By Lemma 30, $\operatorname{cl}(\mathcal{F})_K \subseteq \operatorname{cl}(\mathcal{F}_K)$ and the right hand-side is an essentially finite graph class by Corollary 32. $\square$

5.1. Applications of Theorem 27

To demonstrate the inner workings of condition Item 2 in Theorem 27, we consider the following examples. The first example shows that not even the weakest closure property from Fig. 1 is shared by all homomorphism distinguishing closed families. The second example answers a question from [10, p. 29] negatively: Is the disjoint union closure of the union of homomorphism distinguishing closed families homomorphism distinguishing closed? Finally, the second and third example illustrate that the inclusions in Lemma 4 can be proper.

Example 34. *Let F_1 and F_2 be connected non-isomorphic homomorphically equivalent simple graphs.*

1. *The class $\mathcal{F}_1 \coloneqq \{n(F_1 + F_2) \mid n \geq 1\}$ is homomorphism distinguishing closed and not closed under taking summands.*
2. *For the homomorphism distinguishing closed $\mathcal{F}_2 \coloneqq \{nF_1 \mid n \geq 1\}$, the disjoint union closure of $\mathcal{F}_1 \cup \mathcal{F}_2$ is not homomorphism distinguishing closed,*
3. *Let $\mathcal{F}_3 \coloneqq \{n_1F_1 + n_2F_2 \mid n_1 \geq n_2 \geq 1\}$ and $\mathcal{F}_4 \coloneqq \{n_1F_1 + n_2F_2 \mid n_2 \geq n_1 \geq 1\}$. Then $\operatorname{cl}(\mathcal{F}_3 \cap \mathcal{F}_4) \subsetneq \operatorname{cl}(\mathcal{F}_3) \cap \operatorname{cl}(\mathcal{F}_4)$.*

Proof. For the first example, observe that clearly $\Gamma(\mathcal{F}) = \{F_1, F_2\}$. To verify the assumptions of Theorem 27, let $K \coloneqq n_1F_1 + n_2F_2$ for $n_1, n_2 \geq 0$. It holds that $\operatorname{span}\{\vec{F} \in \mathbb{R}^{\Gamma(\mathcal{F})} \mid F \in \mathcal{F}\} = \{\lambda \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) \mid \lambda \in \mathbb{R}\}$ and thus $\vec{K}$ is in this space only if $n_1 = n_2$. In this case, however, $K \in \mathcal{F}$ and thus $\mathcal{F}$ is homomorphism distinguishing closed. It remains to observe that $F_1 \notin \mathcal{F}$.

In the second example, by Theorem 27, $\mathcal{F}_2$ is homomorphism distinguishing closed. The disjoint union closure of $\mathcal{F}_1 \cup \mathcal{F}_2$ is $\mathcal{F} \coloneqq \{n_1F_1 + n_2F_2 \mid n_1 \geq n_2 \geq 1\}$. Since $\hom(F_1, F_2) > 0$, it holds that $\mathcal{F}_{F_2} = \mathcal{F}$ and hence $\vec{F_2} \in \operatorname{span}\{\vec{F} \in \mathbb{R}^{\Gamma(\mathcal{F})} \mid \vec{F} \in \mathcal{F}_{F_2}\} = \mathbb{R}^{\Gamma(\mathcal{F})}$. However, $F_2 \notin \mathcal{F}$. By Theorem 27, $\mathcal{F}$ is not homomorphism distinguishing closed.

In the third example, by Theorem 27, $\operatorname{cl}(\mathcal{F}_3) = \operatorname{cl}(\mathcal{F}_4) = \{n_1F_1 + n_2F_2 \mid n_1, n_2 \geq 1\}$. However, $\mathcal{F}_3 \cap \mathcal{F}_4 = \{n(F_1 + F_2) \mid n \geq 1\}$, which is homomorphism distinguishing closed by the first example. $\square$

5.2. Complexity of Homomorphism Indistinguishability over Essentially Profinite Graph Classes

From a computational perspective, the central open question on homomorphism indistinguishability regards the complexity and computability of the decision problem $\textsc{HomInd}(\mathcal{F})$. For a fixed graph class $\mathcal{F}$, this problem asks to determine whether two input graphs G and H are homomorphism indistinguishable over $\mathcal{F}$. While $\textsc{HomInd}(\mathcal{TW}_k)$ for the class $\mathcal{TW}_k$ of all graphs of treewidth at most k is polynomial-time [8, 31], $\textsc{HomInd}(\mathcal{P})$ for the class $\mathcal{P}$ of all planar graphs is undecidable [3]. Furthermore, $\textsc{HomInd}(\mathcal{G})$ for the class $\mathcal{G}$ of all graphs coincides with graph isomorphism, which poses well-known complexity-theoretic challenges [35]. These examples illustrate that the complexity-theoretic landscape of the problems $\textsc{HomInd}(\mathcal{F})$ for graph classes $\mathcal{F}$ is diverse and far from being comprehensively understood.

In this section, we show that for all essentially finite classes $\mathcal{F}$, $\textsc{HomInd}(\mathcal{F})$ is in polynomial time. For essentially profinite $\mathcal{F}$, the problem can be arbitrarily hard.

Theorem 35. *Let $\mathcal{F}$ be an essentially finite graph class. Then there exists a finite graph class $\mathcal{F}'$ such that $G \equiv_{\mathcal{F}} H$ if and only if $G \equiv_{\mathcal{F}'} H$ for all simple graphs G and H. In particular, there is a polynomial-time algorithm for* $\textsc{HomInd}(\mathcal{F})$.

Proof. Construct $\mathcal{F}'$ by choosing for every $\Lambda \subseteq \Gamma(\mathcal{F})$ a finite set $F_1^\Lambda, \dots, F_\ell^\Lambda \in \mathcal{F}_L$ such that the $\vec{F}_1^\Lambda, \dots, \vec{F}_\ell^\Lambda \in \mathbb{R}^{\Gamma(\mathcal{F})}$ span the finite-dimensional space $\operatorname{span}\{\vec{F} \in \mathbb{R}^{\Gamma(\mathcal{F})} \mid F \in \mathcal{F}_L\}$ where $L := \coprod_{C \in \Lambda} C$ and taking the union over all graphs $F_1^\Lambda, \dots, F_\ell^\Lambda \in \mathcal{F}_L$ constructed in this way. By construction, $\mathcal{F}' \subseteq \mathcal{F}$. Thus, it suffices to show that $\mathcal{F} \subseteq \operatorname{cl}(\mathcal{F}')$.

Let $K \in \mathcal{F}$ be a graph. By Lemma 28, $K \in \operatorname{cl}(\mathcal{F}')$ if and only if $K \in \operatorname{cl}(\mathcal{F}'_K)$. Let $L := \coprod_{C \in \Lambda} C$ where $\Lambda := \Gamma(K)$. Since L and K are homomorphically equivalent, it holds that $\mathcal{F}'_K = \mathcal{F}'_L$. By construction, $\vec{K} \in \operatorname{span}\{\vec{F}_1^\Lambda, \dots, \vec{F}_\ell^\Lambda\} \subseteq \mathbb{R}^{\Gamma(\mathcal{F})}$ and, by projection, the analogous statement holds in $\mathbb{R}^\Delta$ for $\Delta := \Gamma(\{K, F_1^\Lambda, \dots, F_\ell^\Lambda\})$. By Theorem 29, $K \in \operatorname{cl}(\{F_1^\Lambda, \dots, F_\ell^\Lambda\}) \subseteq \operatorname{cl}(\mathcal{F}')$. ☐

For essentially profinite $\mathcal{F}$, the problem $\textsc{HomInd}(\mathcal{F})$ can be arbitrarily hard. For a set $S \subseteq \mathbb{N}$, write $\textsc{NotIn}(S)$ for the problem of deciding given $n \in \mathbb{N}$ whether $n \notin S$. Here, n is encoded unarily, i.e. the size of an instance is n. Note that (the complement of) any decision problem $L \subseteq \{0,1\}^*$ can be reduced to $\textsc{NotIn}(S)$ for some S at the expense of an exponential increase in the size of the instances. Recall the definition of the graph class $\mathcal{K}^S$ from Eq. (7).

Theorem 36. *For every $S \subseteq \mathbb{N}$, the problem $\textsc{NotIn}(S)$ polynomial-time many-one reduces to $\textsc{HomInd}(\mathcal{K}^S)$.*

The proof of Theorem 36 is build on the following result from [36].

Theorem 37 ([36, Corollary 14])**.** *For every $n \in \mathbb{N}$, there exist graphs G_n and H_n of order $\leq \max\{1, 2(n-1)\}$ such that for all $\ell \in \mathbb{N}$,*

$$\hom(K_\ell, G) \neq \hom(K_\ell, H) \iff \ell = n.$$

Furthermore, G_n and H_n can be constructed in polynomial time in n.

Proof of Theorem 36. Consider the following reduction from $\textsc{NotIn}(S)$ to $\textsc{HomInd}(\mathcal{K}^S)$. Given $n \in \mathbb{N}$, construct G_n and H_n via Theorem 37. Then $G_n \equiv_{\mathcal{K}^S} H_n$ if and only if $n \notin S$. ☐

Although Theorem 36 implies that $\textsc{HomInd}(\mathcal{F})$ for essentially profinite graph classes $\mathcal{F}$ can be arbitrarily hard, the graph classes $\mathcal{K}^S$ arising there are not very well-behaved from a graph-theoretic point of view. In light of Roberson's conjecture [10], minor-closed graph classes are of special interest. The following Lemma 38 shows that considering essentially profinite rather than essentially finite graph classes, in a sense, does not add any value for minor-closed graph classes.

Lemma 38. *Every essentially profinite minor-closed graph class $\mathcal{F}$ is essentially finite.*

Proof. Since the class of all graphs is not essentially profinite, there exists a number $t \in \mathbb{N}$ such that all $F \in \mathcal{F}$ do not contain the clique K_t as a minor. By a weak form of Hadwiger's conjecture [37] proven by Wagner [38], this implies that all $F \in \mathcal{F}$ are 2^{t-1}-colourable. In particular, $\mathcal{F} = \mathcal{F}_{K_{2^{t-1}}}$ is essentially finite. ☐

5.3. Cancellation Laws

The results summarised in Table 1 give necessary conditions for an equivalence relation $\approx$ comparing graphs to be a homomorphism indistinguishability relation over a graph class with certain closure properties. They may be viewed as a step towards an axiomatic characterisation of homomorphism indistinguishability relations, similar to the characterisation [39, 40] of functions $f: \mathcal{G} \to \mathbb{N}$ on the set of all graphs $\mathcal{G}$ which are of the form $f = \hom(-, G)$ for some graph G. Such a characterisation should give sufficient and necessary criteria for an equivalence relation $\approx$ to be a homomorphism indistinguishability relation over some graph class. In this context, exploring abstract properties of equivalence relations comparing graphs and their connection to homomorphism indistinguishability is imperative.

Definition 39. Let K be a simple graph. An equivalence relation $\approx$ comparing graphs *admits K-cancellation* if for all simple graphs G and H it holds that $G \times K \approx H \times K \implies G \approx H$.

Lovász [41] proved that the isomorphism relation $\cong$ admits K-cancellation if and only if K is non-bipartite. The following Lemma 40 shows that K-cancellation of a homomorphism indistinguishability relation $\equiv_{\mathcal{F}}$ depends solely on the homomorphism distinguishing closure of the subclass $\mathcal{F}_K \subseteq \mathcal{F}$ of K-colourable graphs in $\mathcal{F}$. In fact, the proof is based on the observation that the equivalence relation $- \times K \equiv_{\mathcal{F}} - \times K$ coincides with the homomorphism indistinguishability relation $\equiv_{\mathcal{F}_K}$.

Lemma 40. *For a graph class $\mathcal{F}$ and a simple graph K, the following are equivalent:*

1. *$\equiv_{\mathcal{F}}$ admits K-cancellation,*
2. *$\mathcal{F} \subseteq \mathrm{cl}(\mathcal{F}_K)$,*
3. *$\mathrm{cl}(\mathcal{F}) = \mathrm{cl}(\mathcal{F}_K)$.*

Proof. First observe that for graphs G and H the conditions $G \times K \equiv_{\mathcal{F}} H \times K$ and $G \equiv_{\mathcal{F}_K} H$ are equivalent. Indeed, if $G \times K \equiv_{\mathcal{F}} H \times K$ and $F \in \mathcal{F}_K$ then $\hom(F, G) = \hom(F, G \times K)/\hom(F, K) = \hom(F, H)$ and hence $G \equiv_{\mathcal{F}_K} H$. Conversely, if $G \equiv_{\mathcal{F}_K} H$ and $F \in \mathcal{F}$ then $\hom(F, G \times K) = \hom(F, G)\hom(F, K) = \hom(F, H)\hom(F, K) = \hom(F, G \times K)$ since $\hom(F, G) = \hom(F, H)$ if $\hom(F, K) \neq 0$.

By the initial observation, Item 1 is equivalent to the assertion that $G \equiv_{\mathcal{F}_K} H$ implies $G \equiv_{\mathcal{F}} H$ for all graphs G and H. Hence, Items 1 and 2 are equivalent. Since $\mathcal{F} \subseteq \mathrm{cl}(\mathcal{F})$ and $\mathrm{cl}(\mathcal{F}_K) \subseteq \mathrm{cl}(\mathcal{F})$, assertions Items 2 and 3 are equivalent. $\square$

There is a succinct characterisation of essentially profinite graph classes admitting K-cancellation.

Lemma 41. *Let $\mathcal{F}$ be essentially profinite and let K be a simple graph. Then $\equiv_{\mathcal{F}}$ admits K-cancellation if and only if all graphs in $\mathcal{F}$ are K-colourable.*

Proof. By Lemma 31, $\mathrm{cl}(\mathcal{F}_K) \subseteq \mathrm{cl}(\mathcal{F})_K$. By Lemma 40, if $\equiv_{\mathcal{F}}$ admits K-cancellation then $\mathcal{F} \subseteq \mathrm{cl}(\mathcal{F}_K)$ which implies that $\mathcal{F} = \mathcal{F}_K$. The converse is straightforward. $\square$

In particular, if an essentially profinite graph class admits K-cancellation for some simple graph K, then it is essentially finite. Beyond essentially profinite graph classes, it is less clear when $\equiv_{\mathcal{F}}$ admits K-cancellation. If K is bipartite then this depends solely on whether the graphs in $\mathcal{F}$ are K-colourable.

Lemma 42. *For every graph class $\mathcal{F}$ and every bipartite simple graph K, $\mathrm{cl}(\mathcal{F}_K) = \mathrm{cl}(\mathcal{F})_K$. In particular, $\equiv_{\mathcal{F}}$ admits K-cancellation if and only if all graphs in $\mathcal{F}$ are K-colourable.*

Proof. By [10, Lemma 5.8], the class $\mathcal{G}_{K_2}$ of all bipartite graphs is homomorphism distinguishing closed. By Theorem 27, so is $\mathcal{G}_{K_1}$, the family of all graphs which are K_1-colourable, i.e. the edge-less graphs. Finally, the class $\mathcal{G}_{K_0} = \{K_0\}$, the family containing only the empty graph K_0 is homomorphism distinguishing closed. Hence, for $i \in \{0, 1, 2\}$, by Lemma 4,

$$\mathrm{cl}(\mathcal{F}_{K_i}) = \mathrm{cl}(\mathcal{F} \cap \mathcal{G}_{K_i}) \subseteq \mathrm{cl}(\mathcal{F}) \cap \mathrm{cl}(\mathcal{G}_{K_i}) = \mathrm{cl}(\mathcal{F}) \cap \mathcal{G}_{K_i} = \mathrm{cl}(\mathcal{F})_{K_i}.$$

The converse containment follows from Lemma 30. It remains to observe that if K is bipartite then $\mathcal{F}_K$ equals either $\mathcal{F}_{K_0}$, $\mathcal{F}_{K_1}$, or $\mathcal{F}_{K_2}$, depending on whether K is edge-less or empty. $\square$

Under the additional assumption that $\mathcal{F}$ is closed under subdivisions, the following Theorem 43 completes the characterisation of the simple graphs K for which $\equiv_{\mathcal{F}}$ admits K-cancellation. For example, the quantum isomorphism relation $\equiv_{\mathcal{P}}$ [3] admits K-cancellation if and only if K is non-bipartite. Theorem 43 strengthens Lovász' result [41] using an argument from [8].

Theorem 43. *If $\mathcal{F}$ is closed under subdivisions then $\mathcal{F} \subseteq \mathrm{cl}(\mathcal{F}_K)$ for every non-bipartite simple graph K. In particular, $\equiv_{\mathcal{F}}$ admits K-cancellation.*

Proof. The proof is by showing the contrapositive, i.e. for graphs G and H, if $G \not\equiv_{\mathcal{F}} H$, then $G \not\equiv_{\mathcal{F}_K} H$. If there is $F \in \mathcal{F}$ without edges such that $\hom(F, G) \neq \hom(F, H)$ then $G \not\equiv_{\mathcal{F}_K} H$ because $F \in \mathcal{F}_K$. Hence, it may be supposed that there is $F \in \mathcal{F}$ with edge $e \in E(F)$ such that $\hom(F, G) \neq \hom(F, H)$. Since K is non-bipartite, it contains an odd cycle C_{2n+1} for some $n \in \mathbb{N}$. Consider the following claim:

Claim 43.1. *Let F be a simple graph and $n \in \mathbb{N}$. Let F' be obtained from F by replacing every edge by a path on at least $2n + 1$ vertices. Then F' is C_{2n+1}-colourable.*

Proof of Claim. Pick an arbitrary vertex $x \in V(C_{2n+1})$. The C_{2n+1}-colouring of F' is constructed by mapping all vertices from $V(F) \subseteq V(F')$ to x. Let $e \in E(F)$ be an edge. If the number m of vertices introduced in F' by subdividing e is odd, then the resulting path can be mapped to x and one of its neighbours $y \in V(C_{2n+1})$. This is because every path on an odd number of vertices can be mapped to K_2 such that both of its degree-1 vertices are mapped to the same vertex. If m is even, then the path can be mapped to a walk in C_{2n+1} which starts in x, wraps around C_{2n+1} once, and finally alternates between x and one of its neighbours $y \in V(C_{2n+1})$. ⌟

By [8, Lemma 11], there exists a graph F' obtained from F by replacing e by a path of length at least three such that $\hom(F', G) \neq \hom(F', H)$. By repeatedly applying this argument, one obtains a C_{2n+1}-colourable graph F' such that $\hom(F', G) \neq \hom(F', H)$. In particular, F' is K-colourable. Hence, $G \not\equiv_{\mathcal{F}_K} H$. By Lemma 40, this implies that $\mathcal{F} \subseteq \mathrm{cl}(\mathcal{F}_K)$. ☐

The following corollary summarises the results on cancellation properties of homomorphism indistinguishability relations over graph classes closed under subdivisions.

Corollary 44. *Let $\mathcal{F}$ be a graph class which is closed under subdivisions. Let K be a simple graph. Then precisely one of the following holds:*

1. *$\mathcal{F}$ contains a graph which contains a cycle and then $\equiv_{\mathcal{F}}$ admits K-cancellation if and only if K is not bipartite,*
2. *$\mathcal{F}$ contains only acyclic graphs and at least one graph which contains an edge and then $\equiv_{\mathcal{F}}$ admits K-cancellation if and only if K contains an edge,*
3. *$\mathcal{F}$ contains only edgeless graphs and at least one non-empty graph and then $\equiv_{\mathcal{F}}$ admits K-cancellation if and only if K is not empty,*
4. *$\mathcal{F}$ contains only the empty graph and then $\equiv_{\mathcal{F}}$ admits K-cancellation.*

Proof. Since $\mathcal{F}$ is closed under subdivisions, $\equiv_{\mathcal{F}}$ admits K-cancellation for every non-bipartite simple graph K by Theorem 43. If $\mathcal{F}$ contains only acyclic graphs, then all graphs in $\mathcal{F}$ are bipartite and $\equiv_{\mathcal{F}}$ admits K-cancellation for every bipartite simple graph K containing an edge by Lemma 42. If $\mathcal{F}$ contains only edgeless graphs, then $\equiv_{\mathcal{F}}$ admits K-cancellation for every non-empty bipartite simple graph K by Lemma 42. If $\mathcal{F}$ contains only the empty graph, then $\equiv_{\mathcal{F}}$ admits K-cancellation for every bipartite simple graph K by Lemma 42.

If $\mathcal{F}$ contains at least one non-empty graph, then $\equiv_{\mathcal{F}}$ does not admit K_0-cancellation for the empty graph K_0 by Lemma 42. If $\mathcal{F}$ contains at least one edge, then $\equiv_{\mathcal{F}}$ does not admit K-cancellation whenever K does not contain an edge by Lemma 42. If $\mathcal{F}$ contains a graph containing a cycle, then $\mathcal{F}$ contains a non-bipartite graph and hence $\mathcal{F}$ does not admit K-cancellation whenever K is bipartite. ☐

6. Conclusion

The main technical contribution of this work is a characterisation of closure properties of graph classes $\mathcal{F}$ in terms of preservation properties of their homomorphism indistinguishability relations $\equiv_{\mathcal{F}}$, cf. Table 1. In consequence, a surprising connection between logical equivalences and homomorphism indistinguishability over minor-closed graph classes is established. In this way, results from graph minor theory are made available to the study of the expressive power of logics on graphs. Finally, a full classification of the homomorphism distinguishing closed graph classes which are essentially profinite is given. Various open questions of [10] are answered by results clarifying the properties of homomorphism indistinguishability relations and of the homomorphism distinguishing closure.

It is tempting to view the results in Table 1 as instances of a potentially richer connection between graph-theoretic properties of $\mathcal{F}$ and *polymorphisms* of $\equiv_{\mathcal{F}}$, i.e. isomorphism-invariant maps $\mathfrak{f}$ sending tuples of graphs to graphs such that $\mathfrak{f}(G_1, \dots, G_k) \equiv_{\mathcal{F}} \mathfrak{f}(H_1, \dots, H_k)$ whenever $G_i \equiv_{\mathcal{F}} H_i$ for all $i \in [k]$. Recalling the algebraic approach to CSPs, cf. [42], one may ask what structural insights into $\mathcal{F}$ can be gained by considering polymorphisms of $\equiv_{\mathcal{F}}$. More concretely, can closure under topological minors be characterised in terms of some polymorphism?

Acknowledgements. I would like to thank David E. Roberson, Louis Härtel, Martin Grohe, Gaurav Rattan, and Christoph Standke for fruitful discussions.

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