

The local realization of generating series of finite Lie rank*

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Editorial note. Three corrections are made to the source text: Definition 1 explicitly requires convergent vector fields; the diagonal coefficient in Lemma 5(ii) is $I! = \prod_k i_k!$; and the central binomial bound in part a) is $\binom{2p}{p} \leq 4^p$. The resulting estimate $\alpha(2d\alpha C)^p p!$ is unchanged.

1 Introduction

Realization of nonlinear systems by state-space is a classical problem in control theory. This problem has been completely solved by Kalman [10] in the case of linear systems. Similarly, it was solved for bilinear systems (see Brockett [2], d'Alessandro, Isidori, Ruberti [1], Fliess [3], Sussmann [12]). In the general case, let us mention the work of Sussmann [13], Hermann, Krener [7] and Jakubczyk [9]: they assume that the solutions are regular at any time and for any inputs. This restriction lead Fliess to study local realization of nonlinear systems [5].

The present paper deals with the latter subject; there are no new results here, but we give elementary proofs of Fliess' results [5]. He shows that to these systems correspond generating series (which are noncommutative formal power series) which have the property that their Lie rank is finite¹. Then he shows existence and unicity of locally reduced (or minimal) realizations and shows that minimality is equivalent to the two following properties: weak local controllability and weak local observability.

We want to give here a simple proof of these results, especially the proof of unicity, for which Fliess uses in [5] sophisticated results on Lie groups and algebras. The present work extends the “syntactic” approach of series with finite Lie rank, as studied in [5], and of nonlinear systems. Hopefully, this will lead to find a realization, which is minimal (in some sense) and which is given directly by the generating

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¹Bilinear systems correspond to generating series whose rank is finite.

series; this would be analogue to the bilinear case, where the minimal state-space is directly given by the Hankel matrix. Our main tool here is the theorem of Poincaré-Birkhoff-Witt. We treat only the “analytic” case of [5]. The “formal” case is simpler and is obtained by omitting in the sequel everything dealing with convergence or majoration of coefficients.

I want to thank Michel Fliess for encouraging and stimulating discussions on this subject.

2 Generating series of finite Lie rank

We consider a system of the following form

$$\begin{cases} \dot{q}(t) = A_0(q) + \sum_{i=1}^n u_i(t) A_i(q) \\ y(t) = h(q) \end{cases} \quad (1)$$

where the state q belongs to a connected analytic $\mathbb{R}$ -variety Q , where the A_i 's are analytic vector fields and h a real analytic function defined in a neighborhood of the given initial state $q(0)$. The inputs $u_1, \dots, u_n$ are real and piecewise continuous.

By Fliess [4] th.III.2, the output y of system (1) is given by the following formula (for small enough time and inputs):

$$y(t) = h|_{q(0)} + \sum_{\nu \geq 0} \sum_{j_0, \dots, j_\nu \geq 0}^n \left(A_{j_0} \dots A_{j_\nu} h|_{q(0)} \right) \int_0^t d\xi_{j_\nu} \dots d\xi_{j_0} \quad (2)$$

where $|_q(0)$ means evaluation in $q(0)$ and where the iterated integrals $\int d\xi_{j_\nu} \dots d\xi_{j_0}$ are defined by the formulas (u_0 is the constant function equal to 1):

$$\begin{aligned} \int_0^t d\xi_j &= \int_0^t u_j(\tau) d\tau, \text{ and if } \nu \geq 1 \\ \int_0^t d\xi_{j_\nu} \dots d\xi_{j_0} &= \int_0^t \left[u_{j_\nu}(\tau) d\tau \int_0^\tau d\xi_{j_{\nu-1}} \dots d\xi_{j_0} \right] \end{aligned}$$

Actually, the input-output behaviour of system (1) is completely defined by its **generating series** ([4] p.12), which is a noncommutative formal power series in the variables $x_0, \dots, x_n$:

$$g = h|_{q(0)} + \sum_{\nu \geq 0} \sum_{j_0, \dots, j_\nu \geq 0}^n \left(A_{j_0} \dots A_{j_\nu} h|_{q(0)} \right) x_{j_\nu} \dots x_{j_0} \quad (3)$$

Recall that a formal power series in the noncommutative variables $x_0, \dots, x_n$ is a mapping g from the free monoid X^* generated by $X = \{x_0, \dots, x_n\}$ into $\mathbb{R}$, denoted by $w \rightarrow (g, w)$, for any **word** (= element of X^*) w , including the empty word, denoted by 1. The formal series g will also be denoted by

$$g = \sum_{w \in X^*} (g, w) w$$

The set of formal series is denoted by $\mathbb{R}\langle\langle X \rangle\rangle$. Following [5], we give a definition, inspired by (3). First, let us say that a (commutative) formal series in $\mathbb{R}[[q]] = \mathbb{R}[[q_1, \dots, q_d]]$ (where the q_i 's are commutative variables) is **convergent** if it converges in a neighborhood of 0; similarly, we say that a **formal vector field**, that is, an operator of $\mathbb{R}[[q]]$ of the form²

$$A = \sum_{1 \leq k \leq d} \theta_k(q_1, \dots, q_d) \frac{\partial}{\partial q_k}$$

where the θ_k are in $\mathbb{R}[[q]]$, is **convergent**, if the θ_k 's are convergent formal series. Because of the reverse order in (3) of the A 's and the x 's, we let the formal vector fields operate at the right of the formal series.

Definition 1. A formal series $g \in \mathbb{R}\langle\langle X \rangle\rangle$ is **produced differentially** if there exists an integer d , an homomorphism μ from the free monoid into the multiplicative monoid of the endomorphisms of $\mathbb{R}[[q]]$ such that μx is a convergent formal vector field for any x in X , a convergent formal series h in $\mathbb{R}[[q]]$ such that

$$\forall w \in X^*, \quad (g, w) = h(\mu w)|_0$$

(that is, (g, w) is the constant term of the series which is the image of h under the operator μw).

We call the couple (μ, h) a **differential representation** of g , of **dimension** d .

It is now obvious that if g is the generating series of system (1), then it is produced differentially (take local coordinates around $q(0)$ in (3)). Conversely, if a series g is produced differentially, then one may associate to it a system of type (1), whose generating series is g . Thus, **the study of differential representations of series is equivalent to the study of local realizations of systems like (1)**.

Before stating the main result, we need some notations. Denote by $\mathbb{R}\langle X \rangle$ the set of noncommutative **polynomials**, that is, formal series having only a finite number of nonzero coefficients. Then $\mathbb{R}\langle\langle X \rangle\rangle$ is isomorphic to the dual of $\mathbb{R}\langle X \rangle$ (because $\mathbb{R}\langle X \rangle \simeq \mathbb{R}^{(X^*)}$ and $\mathbb{R}\langle\langle X \rangle\rangle \simeq \mathbb{R}^{X^*}$, as vector spaces), with duality

$$\begin{aligned} \mathbb{R}\langle\langle X \rangle\rangle \times \mathbb{R}\langle X \rangle &\rightarrow \mathbb{R} \\ (S, P) &\mapsto (S, P) = \sum_{w \in X^*} (S, w)(P, w) \end{aligned}$$

The set $\mathbb{R}\langle X \rangle$ possesses an associative product, which extends linearly the concatenation of words in X^* . Thus, the algebra $\mathbb{R}\langle X \rangle$ acts naturally at the left and at the right of $\mathbb{R}\langle\langle X \rangle\rangle$, in the following way

$$\begin{aligned} S \circ P &= \sum_{w \in X^*} (S, Pw)w \\ P \circ S &= \sum_{w \in X^*} (S, wP)w \end{aligned} \tag{4}$$

²A formal vector field may also be defined as a derivation of the algebra $\mathbb{R}[[q]]$, which is continuous for the usual topology of $\mathbb{R}[[q]]$.

These two actions are associative (that is, $S \circ PQ = (S \circ P) \circ Q$ and $PQ \circ S = P \circ (Q \circ S)$) and commute each to another (that is $P \circ (S \circ Q) = (P \circ S) \circ Q$).

We denote by $L\langle X \rangle$ the Lie algebra generated by the elements of X : an element of $L\langle X \rangle$ is called a **Lie polynomial**.

Definition 2. The **Lie rank** of a formal series $g \in \mathbb{R}\langle\langle X \rangle\rangle$ is the dimension of the vector space

$$\{P \circ g \mid P \in L\langle X \rangle\}$$

We say that a formal series $g \in \mathbb{R}\langle\langle X \rangle\rangle$ satisfies to the **convergence hypothesis** (C) if: for any Lie polynomials $P_1, \dots, P_q$, there exist constants α and C such that

$$\forall i_1, \dots, i_q \in \mathbb{N}, \quad |(g, P_1^{i_1} \dots P_q^{i_q})| \leq \alpha C^{i_1 + \dots + i_q} i_1! \dots i_q! \quad (\text{C})$$

Note that one may replace $i_1! \dots i_q!$ by $(i_1 + \dots + i_q)!$, because these numbers satisfy to

$$i_1! \dots i_q! \leq (i_1 + \dots + i_q)! \leq q^{i_1 + \dots + i_q} i_1! \dots i_q!$$

Now, we can state the following fundamental result.

Theorem (Fliess [5]). A series $g \in \mathbb{R}\langle\langle X \rangle\rangle$ is differentially produced if and only if its Lie rank is finite and if it satisfies to the convergence hypothesis. In this case, its Lie rank d is equal to the smallest dimension of all its differential representations. If (μ, h) and (μ', h') are two differential representations of dimension d of g (with the same $\mathbb{R}[[q]]$), then there exists a continuous and convergent³ automorphism of $\mathbb{R}[[q]]$ such that $h' = \phi(h)$ and $\phi(k\mu w) = \phi(k)\mu'w$ for any word w and any series k in $\mathbb{R}[[q]]$.

Note that the fact that ϕ is bijection is equivalent to the well-known Jacobian condition

$$\left| \frac{\partial \phi(q_j)}{\partial q_i}(0) \right| \neq 0$$

The third part of this paper is devoted to the proof of the theorem. We need before some definition and results.

On $\mathbb{R}\langle\langle X \rangle\rangle$ is defined another product, the **shuffle**, which is associative and commutative and which will be important in the sequel; this is not a surprise, as the shuffle intervenes already for systems of the form (1): indeed, by Fliess ([4] prop. II.4), if two systems admit the generating series g_1 and g_2 , then the system whose output is the product of the outputs of these two systems admits as generating series the shuffle of g_1 and g_2 .

If w is a word of length $|w| = p$ ($w = y_1 \dots y_p$, $y_i \in X$) and $I \subset \{1, \dots, p\}$, denote by $w|I$ the word $y_{i_1} \dots y_{i_r}$, with $I = \{i_1 < \dots < i_r\}$. Then the shuffle $u \star v$ of two words u and v is defined as

$$u \star v = \sum w(I, J)$$

³that is, for any q_i , $\phi(q_i)$ is convergent and without constant term.

where the sum is extended to all partitions (I, J) of $(1, \dots, |u| + |v|)$ with $\text{Card}(I) = |u|$, $\text{Card}(J) = |v|$ and where the word $w(I, J)$, of length $|u| + |v|$, is defined by

$$w(I, J)|I = u, \quad w(I, J)|J = v$$

Note that there are $\binom{|u|+|v|}{|u|}$ such words.

Example. $y_1 y_2 \star y_2 y_3 = 2y_1 y_2^2 y_3 + y_1 y_2 y_3 y_2 + y_2 y_1 y_2 y_3 + y_2 y_1 y_3 y_2 + y_2 y_3 y_1 y_2$.

The shuffle $S \star T$ of two series in $\mathbb{R}\langle\langle X \rangle\rangle$ is then defined by

$$S \star T = \sum_{u, v \in X^*} (S, u)(T, v) u \star v$$

(extension by linearity and continuity). **From now on, we denote $S \star T$ simply by ST :** there will be no ambiguity because we consider here only the shuffle structure of $\mathbb{R}\langle\langle X \rangle\rangle$ ⁴. However, $\mathbb{R}\langle X \rangle$ will be considered with its concatenation structure.

Lemma 1. *Let $c_p : \mathbb{R}\langle X \rangle \rightarrow \mathbb{R}\langle X \rangle^{\otimes p}$ be the concatenation homomorphism defined by*

$$c_p(x) = x \otimes 1 \cdots \otimes 1 + 1 \otimes x \cdots \otimes 1 + 1 \otimes 1 \cdots \otimes x$$

For any series $S_1, \dots, S_p$ in $\mathbb{R}\langle\langle X \rangle\rangle$ and any polynomial P one has

$$(S_1 \cdots S_p, P) = (S_1 \otimes \cdots \otimes S_p, c_p(P))$$

where the duality between $\mathbb{R}\langle\langle X \rangle\rangle$ and $\mathbb{R}\langle X \rangle$ is extended to $\mathbb{R}\langle\langle X \rangle\rangle^{\otimes p}$ and $\mathbb{R}\langle X \rangle^{\otimes p}$, that is

$$(S_1 \otimes \cdots \otimes S_p, P_1 \otimes \cdots \otimes P_p) = (S_1, P_1) \cdots (S_p, P_p)$$

Proof. It is enough to prove the lemma when the S 's and the P 's are words; in this case, it is a simple consequence of the definition of the shuffle. $\square$

Lemma 2. *If P is a Lie polynomial, then*

$$c_p(P) = P \otimes 1 \cdots \otimes 1 + 1 \otimes P \cdots \otimes 1 + 1 \otimes 1 \cdots \otimes P$$

Proof. This is true if $P = x \in X$. Moreover, if it is true for P and Q , then also for $[P, Q] = PQ - QP$, as is easily verified. So, the lemma follows. $\square$

We shall use the following classical result.

Theorem (Poincaré-Birkhoff-Witt). *Let $P_1, \dots, P_n, \dots$ be a basis of $L\langle X \rangle$. Then the polynomials*

$$P_{i_1}^{j_1} \cdots P_{i_r}^{j_r}, \quad r \geq 0, \quad i_1, \dots, i_r, j_1, \dots, j_r \geq 1, \quad i_1 < \cdots < i_r$$

form a basis of $\mathbb{R}\langle X \rangle$.

For a proof, see [8], corollary C p. 92.

⁴ $\mathbb{R}\langle\langle X \rangle\rangle$ possesses also the noncommutative product which extends the concatenation of words.

3 Proof of theorem

a) Let g be a series having the differential representation (μ, g) of dimension d . We show that the Lie rank of g is $\leq d$ and that g satisfies (C). For the first assertion, we do as in [5] II.a. Let T_i ($1 \leq i \leq d$) be the series

$$T_i = \sum_w \frac{\partial(h\mu w)}{\partial q_i} \Big|_0 w$$

Let P be a Lie element. We extend $\mu : X^* \rightarrow \text{End}(\mathbb{R}[[q]])$ to an algebra homomorphism from $\mathbb{R}\langle X \rangle$ into $\text{End}(\mathbb{R}[[q]])$. Then $\mu(P)$ is a continuous derivation (because derivations form a Lie algebra) of $\mathbb{R}[[q]]$, hence

$$\begin{aligned} (P \circ S, w) &= (S, wP) = h\mu(wP)|_0 = (h\mu w)\mu P|_0 \\ &= \left[\sum_{1 \leq i \leq d} \frac{\partial(h\mu w)}{\partial q_i} (q_i \mu P) \right] \Big|_0 = \sum_i (T_i, w) (q_i \mu P) \Big|_0 \end{aligned}$$

Thus $P \circ S = \sum_i (q_i \mu P)|_0 T_i$ is a linear combination of $T_1, \dots, T_d$.

We show now that g satisfies to (C)⁵. By assumption, the series h and $q_i \mu x_j$ ($1 \leq i \leq d$, $0 \leq j \leq n$) are convergent in a neighborhood of 0. We thus may find constants α and C such these series are all bounded by the same series

$$f = \alpha \sum_r C^r q^r = \alpha(1 - Cq)^{-1}$$

in the sense that the coefficient of $q_1^{i_1} \dots q_d^{i_d}$ is bounded by $\alpha C^{i_1 + \dots + i_d}$. It is then easily shown that $h\mu w$ (w of length p) is bounded by the series $f\Delta^p$, where Δ is the differential operator

$$\Delta = d \frac{\alpha}{1 - Cq} \frac{\partial}{\partial q}$$

A simple computation shows that

$$f\Delta^p = \frac{\alpha(d\alpha C)^p 1 \cdot 3 \cdots (2p-1)}{(1 - Cq)^{2p+1}}$$

Hence, we obtain

$$|(g, w)| = |h\mu w|_0 \leq f\Delta^p|_0 = \frac{\alpha(d\alpha C)^p (2p)!}{2^p p!}$$

As $\binom{2p}{p}$ is bounded by 4^p , we obtain

$$|(g, w)| \leq \alpha(2d\alpha C)^p p!$$

⁵We follow Gröbner [6], chap. 1.

Now let $P_1, \dots, P_q$ be Lie polynomials, $y_1, \dots, y_q$ new letters and $g' \in \mathbb{R}\langle\langle y_1, \dots, y_q \rangle\rangle$ the series having the differential representation (μ', h) with $\mu' y_i = \mu P_i$. The previous paragraph implies that if w is a word of length p in the y 's, then one has an inequality of the form

$$|(g', w)| \leq \beta D^p p!$$

which proves (C), in view of the remark following the definition of (C).

b) We come now to the converse, which is the essential part of the theorem. Let g be a series of Lie rank d and which satisfies to the convergence hypothesis.

Let $L_0 = \{P \in L\langle X \rangle \mid P \circ g = 0\}$. By assumption, L_0 is of finite codimension d in $L\langle X \rangle$. Moreover, it is a sub-Lie-algebra of $L\langle X \rangle$. Let $P_1, \dots, P_n, \dots$ be a basis of $L\langle X \rangle$ such that $P_{d+1}, \dots, P_n, \dots$ is a basis of L_0 . Let $S_1, \dots, S_d$ the series defined by $(S_i, P_j) = \delta_{i,j}$ and $(S_i, P_{i_1}^{j_1} \dots P_{i_r}^{j_r}) = 0$ if $r = 0$ or if $j_1 + \dots + j_r \geq 2$ (use the P-B-W theorem). Then

$$g = \sum_{i_1, \dots, i_d \geq 0} \frac{(g, P_1^{i_1} \dots P_d^{i_d})}{i_1! \dots i_d!} S_1^{i_1} \dots S_d^{i_d} \quad (1)$$

Note that the S_i 's have zero constant term, which ensures that the sum is well-defined. In fact, we shall prove a more general result.

Proposition 1. *Let L_0 be a sub-Lie-algebra of $L\langle X \rangle$ of codimension d . Let $P_1, \dots, P_d$ be a basis of $L\langle X \rangle$ modulo L_0 and $S_1, \dots, S_d$ series without constant term such that $(S_i, P_j) = \delta_{i,j}$ and which vanish on the left ideal $J = \mathbb{R}\langle X \rangle L_0$. Then*

$$J^\perp = \{S \mid (S, P) = 0, \forall P \in J\} = \mathbb{R}[[S_1, \dots, S_d]]$$

Moreover, any $S \in \mathbb{R}[[S_1, \dots, S_d]]$ has a unique expression as series in the S_i 's.

Proof. **1)** We show that $J^\perp$ contains $\mathbb{R}[[S_1, \dots, S_d]]$. As $J^\perp$ is closed for the usual topology of $\mathbb{R}\langle\langle X \rangle\rangle$ and closed for the operation $T \rightarrow T \circ P$ ($P \in \mathbb{R}\langle X \rangle$), because J is a left ideal, it is enough to show that it is also closed for the shuffle. Let S, T in $J^\perp$: it suffices to show that $(ST) \circ w$ vanishes on L_0 for any word w (it will imply that ST vanishes on $X^* L_0$, hence on J).

Lemma 3. *$T \rightarrow T \circ x$ is a derivation for the shuffle.*

By the lemma (which is well-known), $(ST) \circ w$ is a linear combination of series of the form $(S \circ u)(T \circ v)$. As S, T vanish on J , we obtain that $S \circ u, T \circ v$ vanish on L_0 .

Lemma 4. *Let $i > j$, $T_1, \dots, T_i$ series without constant term and $Q_1, \dots, Q_j$ be Lie polynomials. Then*

$$(T_1 \dots T_i, Q_1 \dots Q_j) = 0$$

Proof. The lemma follows from Lemmas 1 and 2: write that $(T_1 \dots T_i, Q_1 \dots Q_j) = (T_1 \otimes \dots \otimes T_i, c_i(Q_1 \dots Q_j))$ and note that $c_i(Q_1 \dots Q_j)$, which is the product from 1 to j of

$$Q_\ell \otimes 1 \dots \otimes 1 + 1 \otimes Q_\ell \dots \otimes 1 + 1 \otimes 1 \dots \otimes Q_\ell$$

is a sum of tensors each of which has a 1 as factor; as $(T_k, 1) = 0$, we obtain Lemma 4. $\square$

By this lemma, the shuffle of two series which vanish on L_0 is still vanishing on L_0 . Hence each $(S \circ u)(T \circ v)$ vanishes on L_0 , and so does also $(ST) \circ w$.

2) We prove that $J^\perp \subset \mathbb{R}[[S_1, \dots, S_d]]$. Let $S \in J^\perp$. We have to find coefficients $a_{i_1, \dots, i_d}$ such that

$$S = \sum_{i_1, \dots, i_d} a_{i_1, \dots, i_d} S_1^{i_1} \dots S_d^{i_d} \quad (2)$$

For $I = (i_1, \dots, i_d)$, let $a(I) = a_{i_1, \dots, i_d}$, $S(I) = S_1^{i_1} \dots S_d^{i_d}$, $P(I) = P_1^{i_1} \dots P_d^{i_d}$, $|I| = i_1 + \dots + i_d$ and $I! = i_1! \dots i_d!$. By the P-B-W theorem, we have to show that both sides of (2) have the same value on any polynomial of the form

$$P_1^{i_1} \dots P_r^{i_r}, \quad r \geq 0, \quad 1 \leq i_1 < \dots < i_r, \quad 1 \leq j_1, \dots, j_r$$

But, if $r > d$, then this polynomial is in J , hence both sides map it to zero. Hence, we have to find coefficients $a(I)$ such that

$$\forall J, \quad (S, P(J)) = \sum_I a(I) (S(I), P(J)) \quad (3)$$

Lemma 5. (i) If $|I| > |J|$ or if $|I| = |J|$ and $I \neq J$, then $(S(I), P(J)) = 0$
(ii) $(S(I), P(I)) = I!$

Proof. If $|I| > |J|$, use Lemma 4. Otherwise, use Lemmas 1 and 2 to prove that

$$(T_1 \dots T_n, Q_1 \dots Q_n) = \sum_{\sigma \in \mathfrak{S}_n} (T_1, Q_{\sigma(1)}) \dots (T_n, Q_{\sigma(n)})$$

which is true for any series T_i without constant term and any Lie polynomials Q_j . $\square$

Lemma 5 shows that (3) is a triangular system of linear equations in the a 's, with $I!$ on the diagonal. Hence, it admits one and only one solution, which proves the proposition. $\square$

(1) is proved by using the fact that in this case one has: $|I| < |J|$ implies $(S(I), P(J)) = 0$ (use Lemmas 1 and 2).

(1) gives almost the differential representation of g . Indeed, g is given in (1) as a commutative series in $S_1, \dots, S_d$, and by Proposition 1, $\mathbb{R}[[S_1, \dots, S_d]]$ is isomorphic to an algebra of commutative formal power series in d variables. We have to define μ and h . We let $h = g$ and define μw ($w \in X^*$) as $T \rightarrow T \circ w$.

By Lemma 3, μx is a continuous derivation, which maps $\mathbb{R}[[S_1, \dots, S_d]]$ into itself (by the proposition). Moreover

$$(g, w) = (g \circ w, 1) = (h\mu w, 1)$$

and the constant term of $h\mu w$ is also the constant term of $h\mu w$ when expressed as a series in the S_i 's (because the latter are without constant term). We still have to show that the operators μx are convergent. The series $P_1 \circ g, \dots, P_d \circ g$ being linearly independent, we may find polynomials $Q_1, \dots, Q_d$ such that

$$(P_i \circ g, Q_j) = \delta_{i,j}$$

Let $T_1, \dots, T_d$ be defined by

$$T_i = g \circ Q_i - (g, Q_i) \quad (4)$$

The T_i 's are without constant term, vanish on J and we have

$$(T_i, P_j) = (g \circ Q_i, P_j) = (g, Q_i P_j) = (P_j \circ g, Q_i) = \delta_{j,i}$$

Hence by the proposition

$$\mathbb{R}[[T_1, \dots, T_d]] = J^\perp = \mathbb{R}[[S_1, \dots, S_d]]$$

As for g , we have relations of the form

$$T_j = \sum_I \frac{(T_j, P(I))}{I!} S(I) \quad (5)$$

Moreover by (4), the T_j 's satisfy to the convergence hypothesis. Thus, by (5), the T_j 's may be written as convergent series in the S_i 's. We use now the following classical result.

Theorem (of implicit functions). *Let $t_1, \dots, t_d$ convergent series in $\mathbb{R}[[s_1, \dots, s_d]]$ without constant term and such that $\mathbb{R}[[s]] = \mathbb{R}[[t]]$. Then each s_i may be written as a convergent series in $t_1, \dots, t_d$.*

By this theorem, each S_i is a convergent series in $T_1, \dots, T_d$. As previously, the series

$$T_j \mu x = T_j \circ x$$

satisfy to (C) and are thus convergent series in the S_i 's; hence, they are also convergent series when expressed as series in the T_i 's. All this shows that (μ, h) is a differential representation of g .

c) Now, let g be a series of Lie rank d and (μ, h) be a differential representation of dimension d of g . We use the notations of paragraph b).

Lemma 6. *The mapping $\eta : \mathbb{R}[[q]] \rightarrow \mathbb{R}\langle\langle X \rangle\rangle$ which maps k onto $\sum_w (k\mu w|_0) w$ is a continuous homomorphism (for the shuffle), such that for any word w one has $\eta(k\mu w) = \eta(k) \circ w$.*

Proof. This lemma is a simple consequence of [4] prop. III. 1. $\square$

Lemma 7. *The mapping $\theta : \mathbb{R}\langle\langle X \rangle\rangle \rightarrow \mathbb{R}\langle\langle X \rangle\rangle$ which maps S onto $\sum_I \frac{(S, P(I))}{I!} S(I)$ is a continuous shuffle homomorphism.*

Proof. In order to prove this lemma, note that if $S_1, \dots, S_n, \dots$ are the series which are the elements of the dual basis of the P-B-W basis constructed on $P_1, \dots, P_n, \dots$, which correspond to $P_1, \dots, P_n, \dots$ then $\mathbb{R}\langle\langle X \rangle\rangle = \mathbb{R}[[S_1, \dots, S_n, \dots]]$ and the mapping of Lemma 7 is just a projection: $S_1 \rightarrow S_1, \dots, S_d \rightarrow S_d, S_n \rightarrow 0$ if $n > d$. $\square$

By Lemma 6, $\eta(\mathbb{R}[[q]])$ contains $g = \eta(h)$ and is closed for the operations $T \rightarrow T \circ w$. Hence, it contains the T_i 's defined by (4), hence also $\mathbb{R}[[T_i]] = \mathbb{R}[[S_i]]$. As the restriction of θ to $\mathbb{R}[[S_i]]$ is the identity, the mapping $\phi = \theta \circ \eta : \mathbb{R}[[q]] \rightarrow \mathbb{R}[[S_i]]$ is surjective. As it is a continuous homomorphism from an algebra of formal power series in d commutative variables into another, ϕ is also injective. We deduce that η is also a bijection $\mathbb{R}[[q]] \rightarrow \mathbb{R}[[S_i]]$: first, η is injective (otherwise $\phi = \theta \circ \eta$ is not); moreover we may find series $k_1, \dots, k_d$ in $\mathbb{R}[[q]]$ such that $q_i \rightarrow k_i$ is a continuous automorphism of $\mathbb{R}[[q]]$ and such that $\phi(k_i) = S_i$. As $\eta(\mathbb{R}[[q]])$ contains $\mathbb{R}[[S_i]]$ and $\mathbb{R}[[q]] = \mathbb{R}[[k_i]]$, we have that S_i is a series in the $\eta(k_i)$'s: $S_i = s(\eta(k_1), \dots, \eta(k_d))$.

Apply θ : then $S_i = \theta(S_i) = s(S_1, \dots, S_d)$, which shows that s has only the term S_i , hence $S_i = \eta(k_i)$. This shows that $\eta(\mathbb{R}[[q]]) = \eta(\mathbb{R}[[k_i]]) = \mathbb{R}[[\eta(k_i)]] = \mathbb{R}[[S_i]]$ and η is a bijection as claimed.

We still have to prove the assertions about convergence (we have already seen that any differential representation of dimension d of g is isomorphic to the one defined by $S_1, \dots, S_d$). By assumption, the series g and the operators $T \rightarrow T \circ x$ of $\mathbb{R}[[S_i]]$ are convergent when expressed as series in the $\eta(q_i)$'s (as h and μx are convergent, when expressed in the q_i 's). Hence the series T_i of (4) are convergent in the $\eta(q_i)$'s. Hence η is a convergent isomorphism from $\mathbb{R}[[q]]$ onto $\mathbb{R}[[T_i]]$. This ends the proof of the theorem.

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