

Max Independent Set Remains NP-hard when Excluding a Planar Induced Minor

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Abstract

We show that there is a fixed planar graph H , namely the 5×5 grid, such that MAX INDEPENDENT SET remains NP-hard in H -induced-minor-free graphs. This refutes the Dallard–Milanič–Štorgel conjecture and a weakening of it by Gartland and Lokshtanov, and by Korhonen.

1 Introduction

Graph classes excluding a planar minor have bounded treewidth [14]. Thus, in them, any problem definable in MSO_2 logic (or its optimization variants) can be solved in polynomial time [5]. This cannot hold in classes excluding a planar *induced* minor (where induced minors are obtained by removing vertices and contracting edges). Indeed, a problem like DOMINATING SET is NP-complete on graphs excluding a 5-vertex path as an induced minor [2] (or, equivalently, as an induced subgraph).

However, no similar direct argument dismisses that possibility for the MAX INDEPENDENT SET problem (MIS for short). This led Dallard, Milanič, and Štorgel to ask whether a polynomial-time algorithm exists for MIS on any planar-induced-minor-free class.

► **Conjecture 1** (Dallard–Milanič–Štorgel [6]). *For every planar graph H , MAX INDEPENDENT SET can be solved in polynomial time on graphs excluding H as an induced minor.*

This question, turned into a conjecture, has garnered a large amount of related work in the past five years. (Let K_t denote the t -vertex clique, $K_{s,t}$ the biclique with s vertices against t vertices, P_t the t -vertex path, C_t the t -vertex cycle, and W_t the wheel obtained from C_t by adding a universal vertex.) Conjecture 1 was shown when H is K_5 minus an edge, $K_{2,t}$, W_4 [6], P_6 [9], or any windmill graph (a matching plus a universal vertex) [4].

A natural relaxation of Conjecture 1 to quasipolynomial time has seen further development.

► **Conjecture 2** (Gartland–Lokshtanov, Korhonen [12], [4, Question 2]). *For every planar graph H , MAX INDEPENDENT SET can be solved in quasipolynomial time on graphs excluding H as an induced minor.*

While Conjecture 1 is already open when $H = P_7$, Conjecture 2 is known when H is any path, any cycle [8], a disjoint union of triangles [3], or even $tC_3 \uplus C_4$ [4].

Korhonen and Lokshtanov have obtained a subexponential algorithm, as a byproduct of balanced separators of size $\mathcal{O}(\sqrt{m})$, where m is the number of edges.

► **Theorem 3** ([13, Corollary 1.2]). *For every planar graph H , MAX INDEPENDENT SET can be solved in $2^{\tilde{\mathcal{O}}_H(n^{2/3})}$ time on n -vertex graphs excluding H as an induced minor.*

Gartland and Lokshtanov have proposed strengthenings of Conjectures 1 and 2 for a family of problems. Fix a positive integer t , a CMSO_2 sentence φ , i.e., a closed formula with universal and existential quantifications over vertices, vertex subsets, and edge subsets,

and counting predicates modulo some fixed integers. The $(\text{tw} \leq t, \varphi)$ -WEIGHTED MAXIMUM INDUCED SUBGRAPH $((\text{tw} \leq t, \varphi)$ -WMIS) takes as input a vertex-weighted graph (G, w) and asks one either to find a set $S \subseteq V(G)$ such that

- $G[S]$ satisfies φ ,
- the treewidth of $G[S]$ is at most t , and
- the sum of weights of vertices in S is maximized subject to the above conditions,

or to conclude that no such set S exists.

Special cases of the $(\text{tw} \leq t, \varphi)$ -WMIS problem include, for instance, MIS, FEEDBACK VERTEX SET, and EVEN CYCLE TRANSVERSAL. It was conjectured that Conjecture 1 more generally holds for $(\text{tw} \leq t, \varphi)$ -WMIS.

► **Conjecture 4** ([7, Conjecture 1.4.2]). *For every planar graph H , positive integer t , and CMSO₂ sentence φ , $(\text{tw} \leq t, \varphi)$ -WEIGHTED MAXIMUM INDUCED SUBGRAPH can be solved in polynomial time on H -induced-minor-free graphs.*

Our contributions. We refute Conjectures 1, 2, and 4 (unless $P = NP$ for Conjectures 1 and 4, and unless $NP \subseteq QP$ for Conjecture 2) by showing the following.

► **Theorem 5.** *MIS is NP-hard in graphs excluding the 5×5 grid as an induced minor.*

The proof is a remarkably simple reduction from MAX CUT on general graphs F .

Look at Figure 1: this intermediate graph G has the strong product of a path with an edge (i.e., a ladder with diagonals) for each *row* (with each path occupying one of two sides), and a biclique between the two sides in each column. It is easy to show that a cycle in G contains two vertices in the same column and on opposite sides (mentally remove all the column bicliques and consider the possible neighbors of a leftmost vertex of the cycle). The joint neighborhood of two such vertices contains the entire biclique and thus separates the left-hand side from the right-hand side. Therefore, every graph with three vertex-disjoint nonadjacent cycles and a path between every pair of cycles avoiding the neighborhood of the third cycle cannot be an induced minor of G (the 5×5 grid has this property; see Figure 4). Indeed, consider the cycle whose neighborhood contains the biclique in between the bicliques of the other two cycles.

We then pick an appropriate induced subgraph G_F of G , where every odd-index column represents a vertex of F , and every edge of F “occupies” two rows. From the rows of edge $e = uv \in E(F)$, we only keep the vertices in columns between that of u and that of v , with the exception of four vertices; see Figure 2. This defines the graph G_F . Any independent set I in G_F defines a bipartition of $V(F)$ where the part of $v \in V(F)$ depends on the side of the column of v that intersects I . This is well-defined due to the column bicliques. Finally, the encoding of $e \in E(F)$ is such that one more vertex can be inserted in the independent set exactly when its endpoints are in distinct parts of the bipartition of $V(F)$; see Figure 5.

If we reduce from MAX CUT on subcubic graphs F , the previous reduction produces graphs with $\mathcal{O}(|V(F)|^2)$ vertices. Classic reductions thus imply the following, under the Exponential-Time Hypothesis (ETH), i.e., the assumption that there is a $\lambda > 1$ such that n -variable 3-SAT cannot be solved in time $\mathcal{O}(\lambda^n)$ [10].

► **Corollary 6.** *Unless the ETH fails, MAX INDEPENDENT SET requires time $2^{\Omega(\sqrt{n})}$ in n -vertex graphs excluding the 5×5 grid as an induced minor.*

Future work. Now, Conjectures 1 and 2 can be turned into dichotomy questions.

► **Question 1.** Which planar graphs H are such that MAX INDEPENDENT SET can be solved in (quasi)polynomial time on graphs excluding H as an induced minor?

In fact, we give a negative answer to Question 1 for a graph simpler than the 5×5 grid: the cycle C_9 (with $V(C_9) = \{1, \dots, 9\}$) with three additional vertices respectively adjacent to 1 and 2, 4 and 5, and 7 and 8.

One can also turn to approximation schemes. The following conjecture is not refuted by the present work.

► **Conjecture 7.** For every planar graph H , MAX INDEPENDENT SET admits a quasipolynomial-time approximation scheme (QPTAS) on graphs excluding H as an induced minor.

Conjecture 7 is actually known to be implied by a structural conjecture of Gartland and Lokshtanov.

► **Conjecture 8** (Gartland–Lokshtanov [7]). For every planar graph H , there is a constant $k := k(H)$ such that every n -vertex H -induced-minor-free graph G admits a set $X \subseteq V(G)$ of size at most k such that the closed neighborhood of X is a balanced separator of G .

A more adventurous (but still open) conjecture is the strengthening of Conjecture 7 to a polynomial-time approximation scheme (PTAS).

2 Hardness of MIS in Planar-Induced-Minor-Free Classes

For any two integers i, j , we use $[i, j]$ as a shorthand for $\{i, i+1, \dots, j-1, j\}$, and $[i] := [1, i]$. We use the standard graph-theoretic notation. For any graph G , $V(G)$ and $E(G)$ denote the vertex set and edge set, respectively, of G . If $S \subseteq V(G)$, then $G[S]$ denotes the subgraph of G induced by S .

We denote by $\alpha(G)$ the *independence number* of G , that is, the size of a largest subset of vertices of G that are pairwise nonadjacent. We write $\text{MaxCut}(G)$ for the number of edges in a maximum-cardinality cut of G .

► **Theorem 5.** MIS is NP-hard in graphs excluding the 5×5 grid as an induced minor.

Proof. We reduce MAX CUT in general graphs to MAX INDEPENDENT SET in graphs excluding a fixed planar induced minor. Let F be any n -vertex m -edge MAX CUT instance. We first build an intermediate graph G with vertex set

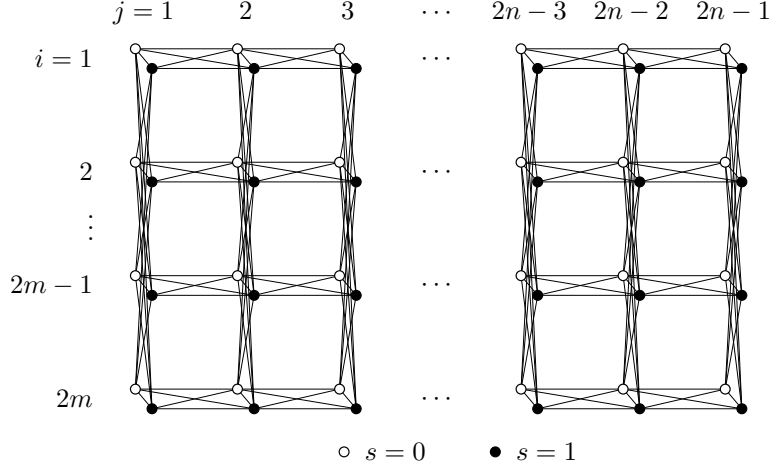
$$\{(i, j, s) \mid i \in [2m], j \in [2n-1], s \in \{0, 1\}\},$$

where we see i as the row index, j as the column index, and s as the side, and edge set

$$\begin{aligned} & \{(i, j, 0)(i', j, 1) \mid i, i' \in [2m], j \in [2n-1]\} \cup \\ & \{(i, j, s)(i, j', s') \mid i \in [2m], j, j' \in [2n-1], |j - j'| = 1, s, s' \in \{0, 1\}\}; \end{aligned}$$

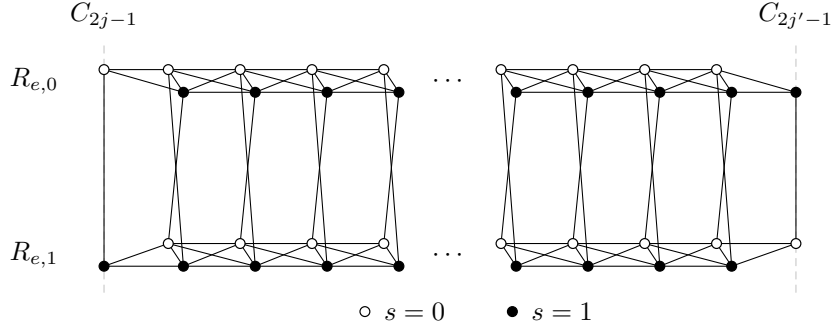
see Figure 1.

We arbitrarily order the vertices of F : $v_1, v_2, \dots, v_n$. We set $C_j := \{(i, j, s) \mid i \in [2m], s \in \{0, 1\}\}$ to be the j th column. We see C_{2j-1} as the column of v_j , for every $j \in [n]$. The columns of even index do *not* represent a vertex of F . Similarly, a *row* is any set $\{(i, j, s) \mid j \in [2n-1], s \in \{0, 1\}\}$ with $i \in [2m]$. We reserve two rows for each edge $e \in E(F)$, and denote them by $R'_{e,0}$ and $R'_{e,1}$. For convenience, we denote by $e^0, e^1 \in [2m]$ the row indices of $R'_{e,0}, R'_{e,1}$, respectively.



■ **Figure 1** The intermediate graph G .

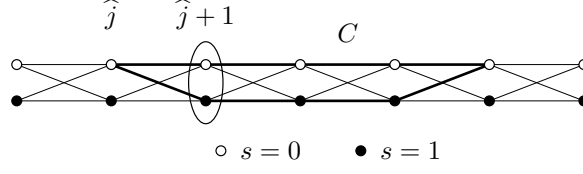
We finally define the following induced subgraph G_F of G . For every edge $e = v_j v_{j'} \in E(F)$ (with $j < j'$), we only keep from $R'_{e,0}$ the vertices $(e^0, 2j-1, 0)$, $(e^0, 2j'-1, 1)$, and all the vertices (e^0, j'', s) with $2j-1 < j'' < 2j'-1$, $s \in \{0, 1\}$, and from $R'_{e,1}$ the vertices $(e^1, 2j-1, 1)$, $(e^1, 2j'-1, 0)$, and all the vertices (e^1, j'', s) with $2j-1 < j'' < 2j'-1$, $s \in \{0, 1\}$. We denote by $R_{e,0} \subset R'_{e,0}$ and $R_{e,1} \subset R'_{e,1}$ the obtained subsets. This completes the construction; see Figure 2. We denote by S_0 the set of vertices of G_F whose last coordinate is 0 (on the side $s = 0$), and by S_1 the set of vertices of G_F on the side $s = 1$.



■ **Figure 2** The restriction of G_F to the two rows reserved for $e = v_j v_{j'}$.

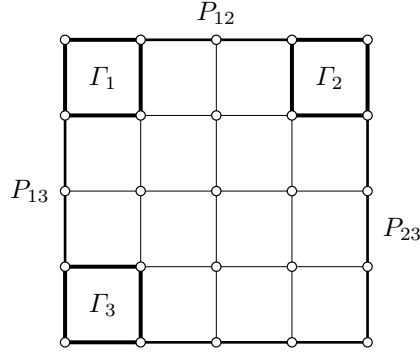
▷ **Claim 9.** Graph G excludes the 5×5 grid as an induced minor.

Proof. We first claim that every cycle of G contains two (adjacent) vertices in the same column but on different sides, i.e., of the form $(i, j, 0)$ and $(i', j, 1)$. Indeed, if one removes from G all the edges with endpoints in the same column but on different sides, the resulting graph G' is the disjoint union of $2m$ graphs Q_{2n-1} obtained from two $(2n-1)$ -vertex paths P_{2n-1} by making the j th vertex of the first path adjacent to the $(j-1)$ st and $(j+1)$ st of the other path (when these vertices exist). In particular, consider a *leftmost* vertex $(\hat{i}, \hat{j}, s)$ (i.e., minimizing $\hat{j}$) of a cycle C that does not contain a pair $(i, j, 0), (i', j, 1)$. Its two neighbors on C must be $(\hat{i}, \hat{j}+1, 0)$ and $(\hat{i}, \hat{j}+1, 1)$; these two vertices form such a pair. See Figure 3 for an illustration.



■ **Figure 3** The leftmost-vertex argument in (a connected component of) G' .

Let H be any graph with three vertex-disjoint and nonadjacent cycles $\Gamma_1, \Gamma_2, \Gamma_3$ such that there is a path between any two cycles in the non-neighborhood of the third cycle. Note that the 5×5 grid is such a graph H ; see Figure 4.



■ **Figure 4** Three appropriate cycles in the 5×5 grid.

We claim that no such graph H is an induced minor of G . The subgraph of G induced by the union U_a of the branch sets representing $V(\Gamma_a)$ (for any $a \in [3]$) has to contain a cycle. By the first paragraph, there are three column indices $j_1, j_2, j_3 \in [2n - 1]$ such that U_a contains a pair of the form $(\cdot, j_a, 0), (\cdot, j_a, 1)$ for each $a \in [3]$. Without loss of generality, let us assume that $j_1 \leq j_2 \leq j_3$. Since the cycles $\Gamma_1, \Gamma_2, \Gamma_3$ are nonadjacent, we further have $j_1 < j_2 < j_3$. Finally, note that, for any $i, i' \in [2m]$, $N_G(\{(i, j_2, 0), (i', j_2, 1)\}) \supseteq C_{j_2}$ and that the removal of C_{j_2} disconnects C_{j_1} from C_{j_3} . This contradicts the assumption that there is a path between Γ_1 and Γ_3 in the non-neighborhood of Γ_2 . $\triangleleft$

In particular, the induced subgraph G_F of G excludes the 5×5 grid as an induced minor. We finish the proof by showing that computing the independence number of G_F would solve MAX CUT for the arbitrary graph F .

▷ **Claim 10.** $\alpha(G_F) = \text{MaxCut}(F) + 2 \sum_{v_j v_{j'} \in E(F), j < j'} j' - j.$

Proof. We first show that

$$\alpha(G_F) \geq \text{MaxCut}(F) + 2 \sum_{v_j v_{j'} \in E(F), j < j'} j' - j.$$

Let (A_0, A_1) be a bipartition of $V(F)$ realizing the maximum cut. We identify A_0 (resp. A_1) with the side $s = 0$ (resp. $s = 1$). We build the independent set I as follows.

For every $v_j \in A_0$ and every incident edge $e = v_j v_{j'} \in E(F)$, add to I the unique vertex of $(R_{e,0} \cup R_{e,1}) \cap C_{2j-1} \cap S_0$. Note that this vertex is in $R_{e,0}$ if $j < j'$ and in $R_{e,1}$ if $j' < j$.

Symmetrically, for every $v_j \in A_1$ and every incident edge $e = v_j v_{j'} \in E(F)$, add to I the unique vertex of $(R_{e,0} \cup R_{e,1}) \cap C_{2j-1} \cap S_1$.

At this stage, for every $e = v_j v_{j'} \in E(F)$ with, say, $j < j'$, I intersects $R_{e,0} \cup R_{e,1}$ at exactly two vertices: one in C_{2j-1} and one in $C_{2j'-1}$. Say that the former (the one in C_{2j-1}) is in $R_{e,s}$, and that the latter is in $R_{e,s'}$, with $s, s' \in \{0, 1\}$. We add the following vertices of $R_{e,0} \cup R_{e,1}$ to the independent set I :

- for every $v_{j''} \in A_0$ with $j'' \in [j+1, j'-1]$, we add the vertex $(e^s, 2j''-1, 0)$;
- for every $v_{j''} \in A_1$ with $j'' \in [j+1, j'-1]$, we add the vertex $(e^s, 2j''-1, 1)$;
- for every $j'' \in [j, j'-2]$, we add $(e^{1-s}, 2j'', 0)$ to I ;
- in addition, if $s = s'$, we add $(e^{1-s}, 2j'-2, 0)$ to I .

(Note that for the last two items, we could also have consistently picked the corresponding vertices in S_1 . In fact, we could have kept only a single vertex at the intersection of a row with a column of even index, but we decided against that to simplify the description of G .) This finishes the construction of I . It is indeed an independent set as we observed the following rules:

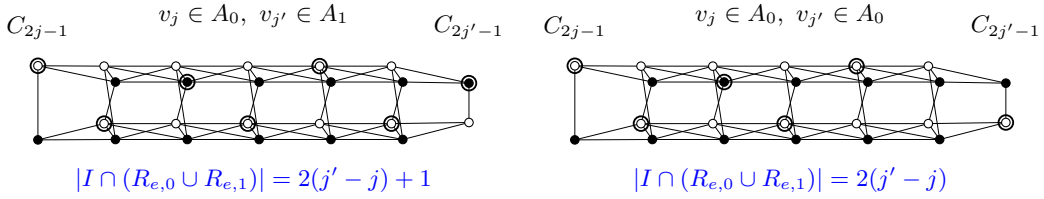
- in every column, we only picked vertices from one side, and
- we never picked two vertices in consecutive columns of the same row.

We now determine the size of I . For every $e = v_j v_{j'} \in E(F)$ with $j < j'$, $I \cap (R_{e,0} \cup R_{e,1})$ contains exactly one vertex in every column intersected by $R_{e,0} \cup R_{e,1}$ except possibly $C_{2j'-2}$, in which it contains at most one vertex. Thus

$$|I \cap (R_{e,0} \cup R_{e,1})| \in \{2(j'-j), 2(j'-j) + 1\}.$$

Furthermore, if e is in the cut (A_0, A_1) , then $s = s'$ and $|I \cap (R_{e,0} \cup R_{e,1})| = 2(j'-j) + 1$. See Figure 5 for an illustration. Therefore,

$$|I| = \text{MaxCut}(F) + 2 \sum_{v_j v_{j'} \in E(F), j < j'} j' - j, \text{ as desired.}$$



■ **Figure 5** Example of $I \cap (R_{e,0} \cup R_{e,1})$ (circled vertices) when v_j and $v_{j'}$ are on opposite sides of the bipartition (A_0, A_1) (left) and when they are on the same side (right).

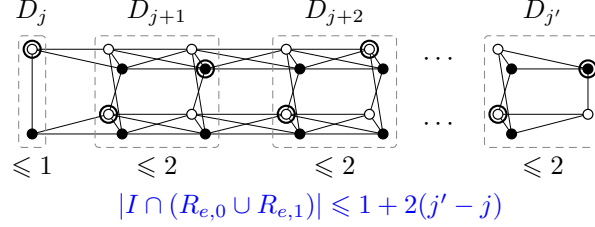
We now show that

$$\alpha(G_F) \leq \text{MaxCut}(F) + 2 \sum_{v_j v_{j'} \in E(F), j < j'} j' - j.$$

We fix some independent set I of G_F . In every column C_{2j-1} of v_j , I can only contain vertices from one side (because of the biclique between $S_0 \cap C_{2j-1}$ and $S_1 \cap C_{2j-1}$). The set I thus defines a partition (A_0, A_1) of $V(F)$ where $v_j \in V(F)$ is in A_0 if $I \cap C_{2j-1} \subseteq S_0$, and $v_j \in A_1$ if instead $I \cap C_{2j-1} \subseteq S_1$. For every $e = v_j v_{j'} \in E(F)$ with $j < j'$, we partition $R_{e,0} \cup R_{e,1}$ into $D_j, D_{j+1}, \dots, D_{j'-1}, D_{j'}$, where

- $D_j := C_{2j-1} \cap (R_{e,0} \cup R_{e,1})$, and
- for every $j'' \in [j+1, j']$, $D_{j''} := (C_{2j''-2} \cup C_{2j''-1}) \cap (R_{e,0} \cup R_{e,1})$.

Observe that I can contain at most one vertex in D_j , and at most two vertices in each $D_{j''}$ with $j'' \in [j+1, j']$. This implies that $|I \cap (R_{e,0} \cup R_{e,1})| \leq 1 + 2(j' - j)$; see Figure 6. Furthermore, if v_j and $v_{j'}$ are on the same side of (A_0, A_1) , one can observe



■ **Figure 6** The partition $D_j, \dots, D_{j'}$ and an upper bound on $|I \cap (R_{e,0} \cup R_{e,1})|$.

that $|I \cap (R_{e,0} \cup R_{e,1})| \leq 2(j' - j)$.

We conclude that $|I| \leq \text{MaxCut}(F) + 2 \sum_{v_j v_{j'} \in E(F), j < j'} j' - j$. ◀

One can easily compute $2 \sum_{v_j v_{j'} \in E(F), j < j'} j' - j$ in polynomial time. So we conclude that solving MIS on G_F is as hard as solving MAX CUT on F . ◀

We obtain the following corollary by reducing from MAX CUT on subcubic graphs.

► **Corollary 6.** *Unless the ETH fails, MAX INDEPENDENT SET requires $2^{\Omega(\sqrt{n})}$ time in n -vertex graphs excluding the 5×5 grid as an induced minor.*

Proof. By the Sparsification Lemma [11], under the ETH, N -variable 3-SAT-B for some constant B (3-SAT where each variable appears at most B times) has no $2^{o(N)}$ -time algorithm. There is a folklore linear reduction from 3-SAT-B to MAX 3-SAT-3 (MAX 3-SAT where each variable appears at most three times). Composing the reductions in [1, Theorems 4.1 and 4.2] gives a linear reduction from MAX E3-SAT-3 to MAX CUT on subcubic graphs. We finally observe that applying the reduction in the proof of Theorem 5 to subcubic MAX CUT instances F produces MIS instances with $n \leq 12|V(F)|^2$ vertices, so $n = \mathcal{O}(N^2)$. ◀

AI disclosure.

The reduction of Theorem 5 was suggested by GPT-6 Pro after, embarrassingly, a mere 23-minute think. We simplified and improved the original complicated argument that the constructed instances exclude the 20×20 grid as an induced minor. The write-up is entirely due to the authors.

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